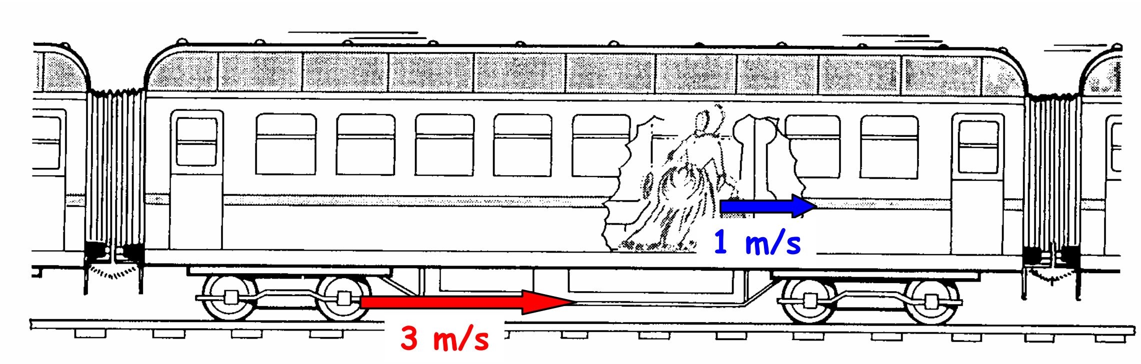
Galilean Relativity



Q. What is the women velocity?

A. With respect to whom?

Frames of Reference:

A frame of reference is a set of coordinates (for example x, y & z axes) with respect to whom any physical quantity can be determined.

Inertial Frames of Reference:

- The inertia of a body is the resistance of changing its state of motion.

- Uniformly moving reference frames (e.g. those considered at 'rest' or moving with constant velocity in a straight line) are called *inertial reference frames*.

- Special relativity deals only with physics viewed from inertial reference frames.

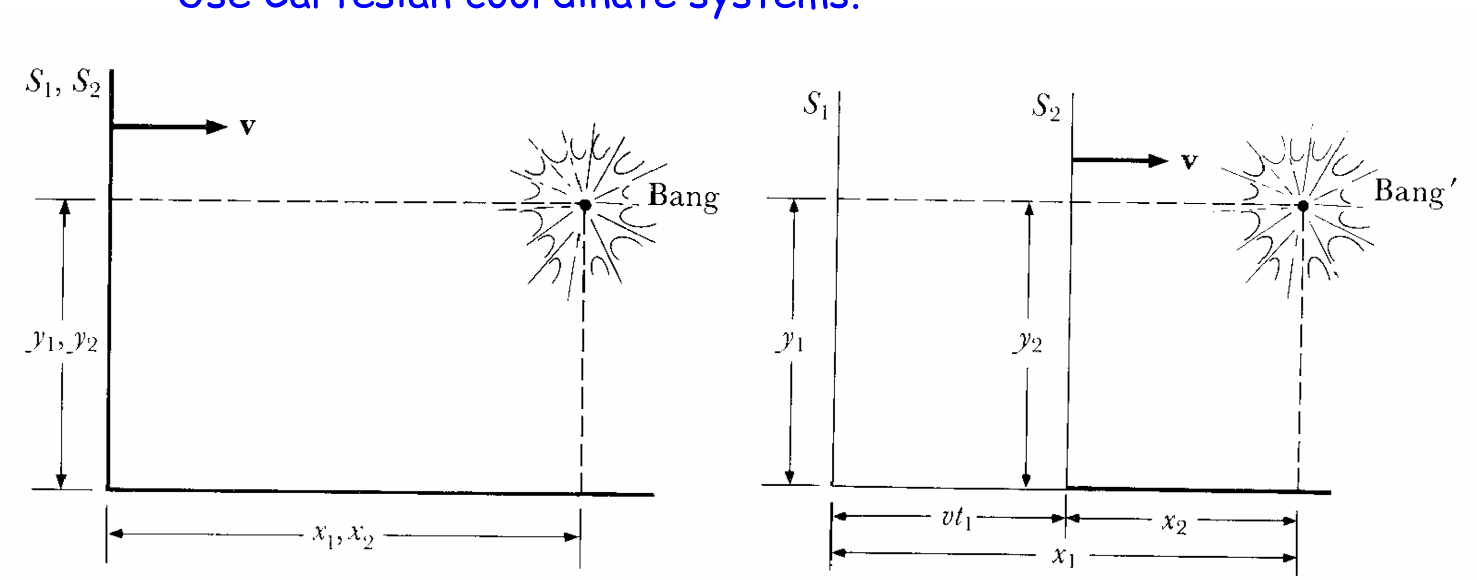
- If we can neglect the effect of the earth's rotations, a frame of reference fixed in the earth is an inertial reference frame.

Galilean Coordinate Transformations:

For simplicity:

- Let coordinates in both references equal at ($t = 0$).

- Use Cartesian coordinate systems.



$t_1 = t_2 = 0$ $t_1 = t_2$

At ($t_1 = t_2$) *Galilean Coordinate Transformations* are:

$$\boxed{\begin{aligned} x_2 &= x_1 - vt_1 \\ y_2 &= y_1 \\ z_2 &= z_1 \end{aligned}} \quad \text{or} \quad \boxed{\begin{aligned} x_1 &= x_2 + vt_2 \\ y_1 &= y_2 \\ z_1 &= z_2 \end{aligned}}$$

Recall *v* is constant, differentiation of above equations gives *Galilean velocity Transformations:*

$$\frac{dx_2}{dt_2} = \frac{dx_1}{dt_1} - v \qquad \frac{dx_1}{dt_1} = \frac{dx_2}{dt_2} - v$$

$$\frac{dy_2}{dt_2} = \frac{dy_1}{dt_1} \qquad \frac{dy_1}{dt_1} = \frac{dy_2}{dt_2}$$

$$\frac{dz_2}{dt_2} = \frac{dz_1}{dt_1} \qquad \text{and} \qquad \frac{dz_1}{dt_1} = \frac{dz_2}{dt_2}$$

or

$$\boxed{v_{x2} = v_{x1} - v} \quad \text{and} \quad \boxed{v_{x1} = v_{x2} + v}$$

Similarly, *Galilean acceleration Transformations:*

$$\boxed{\boldsymbol{a}_2 = \boldsymbol{a}_1}$$

Physics before Relativity

Classical physics was developed between about 1650 and 1900 based on:

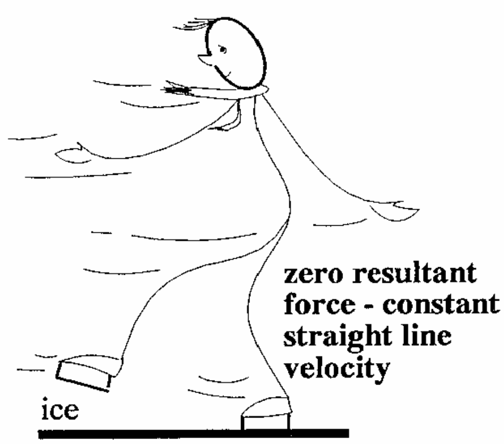
* Idealized mechanical models that can be subjected to mathematical analysis and tested against observation.

* A vast amount of observational data concerning electricity and magnetism that been built up.

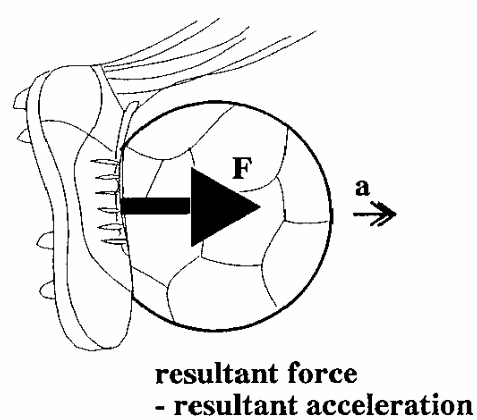
Newtonian Mechanics

1- Newton's Laws of Motion:

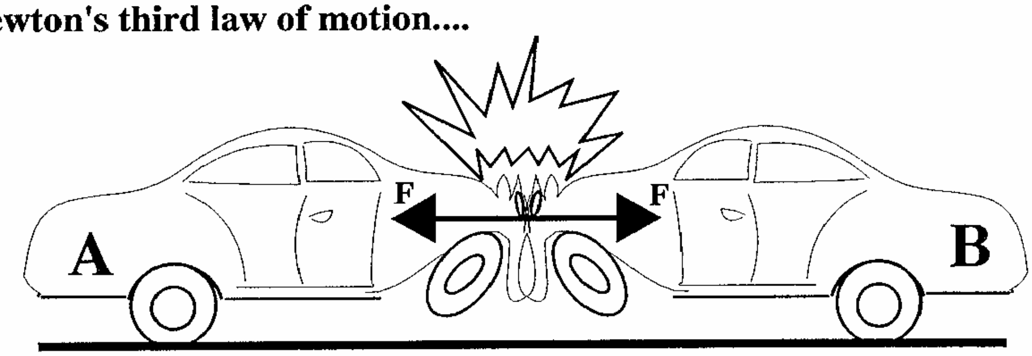
Newton's first law of motion....



Newton's second law of motion....



Newton's third law of motion....



Force of A on B is equal and opposite to the force of B on A.

- Newton's laws are ***only applicable*** *at inertial reference frames.*

- According to Galilean transformations, Newton's laws are invariant at any *inertial reference frame.*

In other words, *the mechanical movement of a particle is exactly the same at two different reference frames, IF both of them are inertial* (i.e. one moves with constant velocity w.r.t the other & vice versa).

- There is no mechanical experiment by which one can distinguish whether a system is at rest or is moving with a constant speed in a straight line (Galilean relativity).

2- Newton's Law of Gravity:

Bodies close to the Earth's surface accelerate downwards when released. Applying the ***first law of motion*** Newton argued that this represented the existence of a resultant force toward the Earth's centre. ***The third law*** implied that a force of equal magnitude but opposite direction would act on the Earth. He assumed this arose from a *mutual attraction between the masses of the two bodies*. He called this attraction GRAVITY and assumed that any massive particle (m_1)

exerts an attractive force (F) on any other massive particle (m_2) lies a distance (r) away, according to the following relation:

$$\boxed{F = -\frac{Gm_1m_2}{r^2}}$$

Where G is called *the universal gravitational constant* that = $6.67\times10^{-11}\ \mathrm{N\ m^2\ kg^{-2}}$

Useful Remarks:

1- Newton stated the existence of an

Absolute

Space **Time** **Mass.**

I.e. Absolute rest.

2- Newton used the word (Mass) in two *apparently* different ways:

Mass

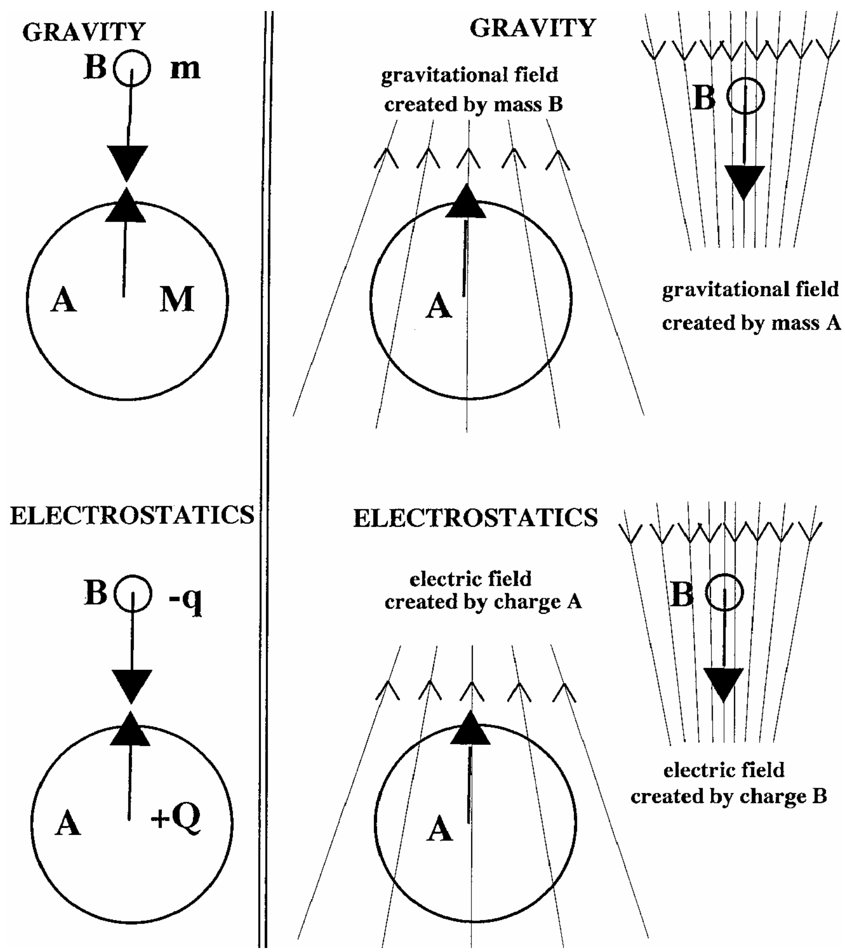
As the inertia of a body (i.e. resists moving)

As a measure of the gravitational force.

Maxwell's Electromagnetism

Faraday & the Concept of Field:

The idea of an 'action-at-a-distance' was held until ***Faraday*** introduced the concept of ***Field*** in (1861). According to his model: charge A creates a field in the space, and charge B - that placed at some point in space- undergoes a force from the field at that point.



Maxwell's theory & Speed of Light:

* For the field theory to be really useful it must account for all the known experimental laws of electromagnetism.

* In **1861** Maxwell succeeded in generating a set of four equations which describe the behavior of fields in all circumstances.

* In Maxwell's theory the electromagnetic field is contains of rapid vibrations of both the electric and magnetic fields. These electromagnetic waves travel at a speed ($c = 1/(\varepsilon_0\mu_0)^{1/2}$), where ε_0 and μ_0 are the permittivity and permeability of free space respectively. Using the data available at the time, Maxwell obtained a velocity of **310,740,000 m/s**.

* Maxwell found that the speed of propagation of an electromagnetic field is approximately that of the **speed of light** obtained by astronomical measurements. He proposed then, that **Light is one of a family of electromagnetic waves**. This was considered as one of the great achievements of 19th century physics.

* The Ether was suggested as a medium in which electromagnetic waves can propagate.

Between Mechanics & Electromagnetism

By 1900 the classical world-view was well established through two main fields: *Mechanics &* Electromagnetism.

But..

All attempts to complete physics by unifying electromagnetism and mechanics failed.

Why?

- According to the Galilean transformations, ***c is not invariant***. Hence, electromagnetic effects ***will not be the same*** for different inertial observers.

I.e. Maxwell's equations ***are not conserved*** by the Galilean transformations, although Newton's laws are.

This fact leads us to one of the followings:

1. Galilean relativity exists both for mechanics and for electromagnetism, but the laws of electromagnetism as given by Maxwell are not correct.

If this is ☑**:** we must be able to perform experiments show deviations from Maxwell's laws.

2. Galilean relativity exists only for mechanics, but not for electromagnetic laws. That is, in electromagnetism there is an *absolute inertial frame* (**the ether**).

If this is ☑: we would be able to locate the ether frame experimentally.

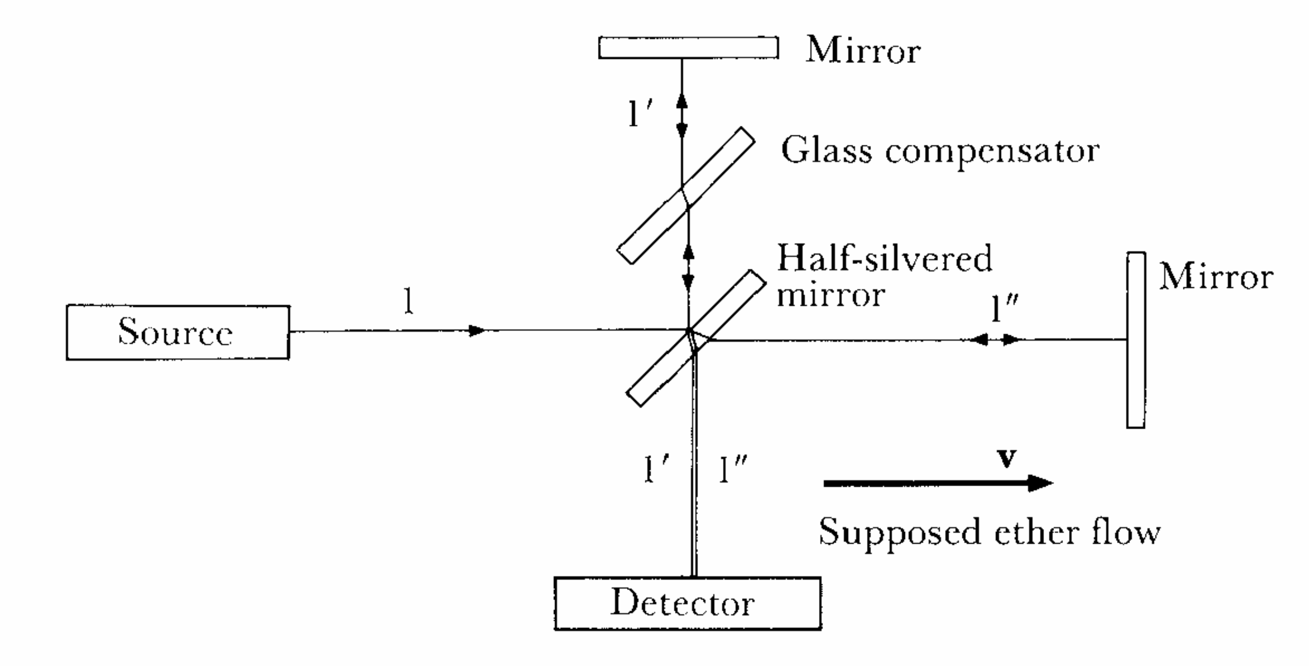
3. Galilean relativity is suitable only for mechanical laws but not for Maxwell's laws.

If this is ☑: *the correct transformation laws would not be the Galilean ones,* but some other ones which are consistent with both mechanics and electromagnetism.

The Michelson-Morley Experiment

One of the experiments designed to measure the speed of the earth through the ether (or to locate the absolute frame) was performed by Michelson and Morley in 1887. It has been done since with greater and greater accuracy in many different versions, but the general agreement was always the same.

The apparatus used in the Michelson-Morley experiment was the **Michelson interferometer**. In this device, monochromatic (one wavelength) light from a source is split into two separate beams. These beams travel two different optical paths and then come back together to interfere either constructively or destructively.



If the earth were moving through the ether, the device could be aligned with the source "upstream" as the ether flowed by.

Ray **I'** would move back and forth across the "ether river" and ray **I''** would move downstream and upstream. If the arms of the interferometer have equal optical lengths **L**, the difference in times for rays **I'** and **I''** will be

$$\Delta T = T'' - T' = \frac{2L/c}{\left(1 - v^2/c^2\right)} - \frac{2L/c}{\sqrt{1 - v^2/c^2}}$$

This difference in times will cause a phase difference and a certain interference pattern with light and dark fringes. In other words if the delay can be measured it will tell us the *Earth's speed w.r.t the ether*.

Since the actual direction of the Earth's motion through the ether was unknown, Michelson and Morley then carefully rotated the device by 90°. This rotation should change the times for rays **I'** and **I''** and then change the interference pattern. Although the expected change was nearly 100 times the sensitivity of their apparatus, ***no shift of the pattern (within experimental error) was discovered***.

Therefore:

The experimenters *could not* find the supposed **ether**.

And this led to an important fact about reference frames:

There is no such thing as an absolute frame of reference in our universe.

Special Theory of Relativity

Special ⟹ means that the theory applies only to ***inertial*** *reference frames*.

Theory ⟹ means that the concept has been confirmed by many different experiments.

Relativity ⟹ means there is **no absolute** frame of reference and hence, any measured values must be **relative.**

Einstein's postulates

Einstein based his *special theory of relativity* on two fundamental postulates:

❶ The principle of relativity:

Gone with the stationary ***ether*** was the idea of ***an absolute*** frame of reference. All motion is relative, not to any stationary system in the universe, but to selected frames of reference.

Thus, for a passenger on a train with no windows, there would be no way to determine whether the train is moving with uniform velocity or is at rest.

This is the first of Einstein's postulates:

All laws of physics have the same mathematical form in all inertial reference frames.

This means:

- There is no preferred frame of reference.
- There is no a physical experiment, *mechanical, electrical or optical* can be performed to determine our state of uniform motion.
- Galilean transformations are not correct for all laws of physics.

❷ The constancy of the speed of light:

Q- "What would a light beam look like if you traveled along beside it?"

A- According to classical physics, the beam would be **at rest** to such an observer.

The more Einstein thought about this, the more convinced he became that one could not move with a light beam. He finally came to the conclusion that no matter how close a person comes to the speed of light, he would still measure the light at $c=3\times10^8$ **m/s**.

This was the second postulate in his special theory of relativity:

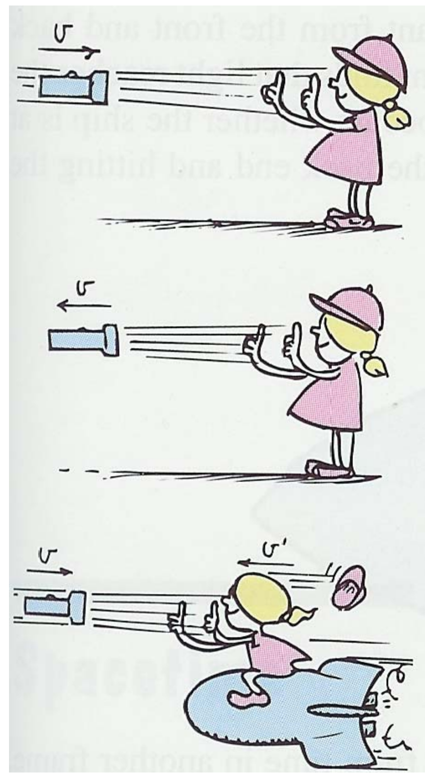
The speed of light in a vacuum has the same measured value (c) in all inertial reference frames.

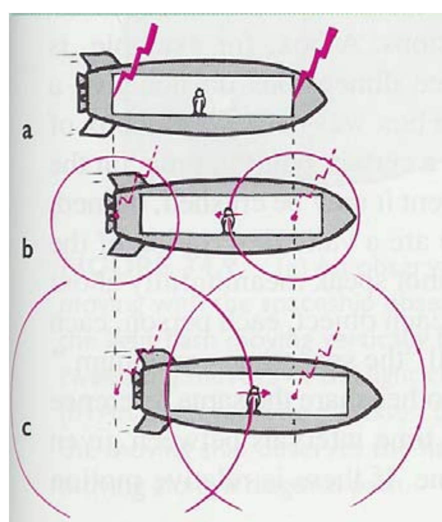
This means:

- The speed of light is invariant.
- The classical idea that space & time are **independent** had to be rejected. (I.e. there is should be a relationship between space & time).
- As a consequence of Einstein's 2nd postulate , is ***the concept of Non-Simultaneity***.

This concept states that:

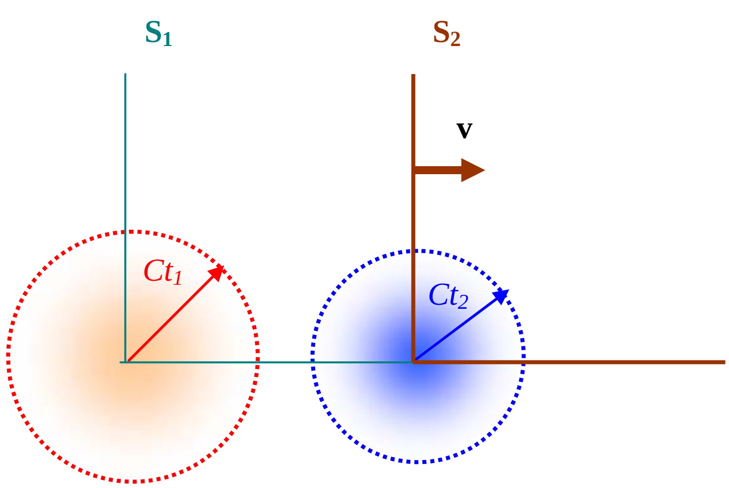
Two events that are simultaneous in one frame of reference ***need not be*** *simultaneous in a frame moving relative to the first frame.*





Lorentz Transformations

- This transformation derives its name from the Dutch physicist **Hendrik Lorentz** (1853-1928).
- Unlike Galilean transformations, Lorentz transformations involve a change of *spatial distance* and a change of *time interval* between two inertial systems. I.e. they are **space-time** transformations.
- Suppose that the coordinate system S_2 is moving with constant velocity v along the x-axis of the coordinate system S_1 , where $y_2 = y_1$ and $z_2 = z_1$.



- Suppose that at $t_1 = t_2 = 0$ a point source of light at the common origin sends out a spherical pulse of light.

- Since c is **a constant** for all observers in both S_1 and S_2 and is the same in all directions, all observers in both frames of reference must detect a spherical wavefront expanding from their origin.

- Since the equation of a sphere is $x^2 + y^2 + z^2 = r^2$ and r, the radius, equals ct, we can write

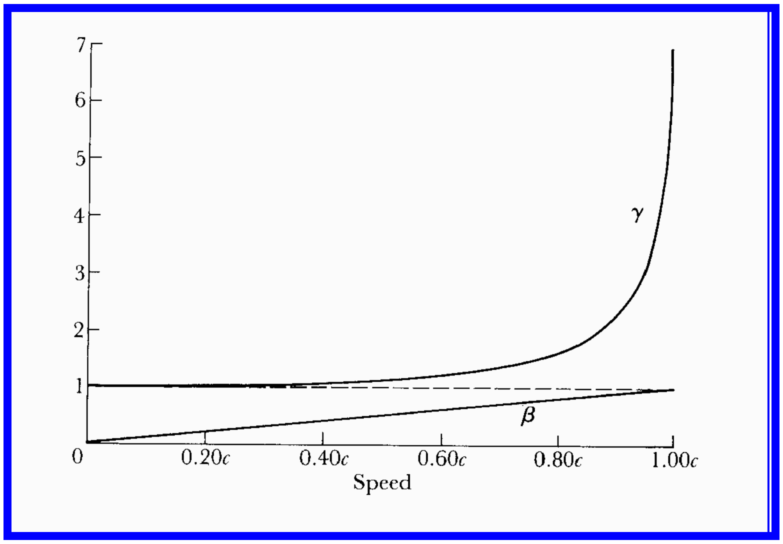
$$\boxed{\begin{aligned} x_1^2 + y_1^2 + z_1^2 - c^2 t_1^2 &= 0 \\ x_2^2 + y_2^2 + z_2^2 - c^2 t_2^2 &= 0 \end{aligned}}$$

- It is easy to see that Galilean transformations will not satisfy both these equations (recall : $x_1 = x_2 + vt$).

- Lorentz derived some formulas that can satisfy both above equations. Those formulas known as ***Lorentz transformations***, and they are:

$$\left.\begin{aligned} x_2 &= \gamma(x_1 - vt_1) \\ y_2 &= y_1 \\ z_2 &= z_1 \\ t_2 &= \gamma\left(t_1 - \frac{v}{c^2}x_1\right) \end{aligned}\right\} \quad \text{where} \quad \begin{aligned} \gamma &= \frac{1}{\sqrt{1-\beta^2}} \\ \beta &= \frac{v}{c} \end{aligned}$$

Note that γ is always greater than or equal to 1 for β less than or equal to 1.



The Correspondence Principle

The *correspondence principle* states that:

Any new theory in physics must reduce to its corresponding well-established classical theory in the situations for which the classical theory is valid.

Or, simply:

new theory + old one must correspond.

Since the physics of Galileo and Newton was experimentally established for objects that moved at speeds much less than the speed of light. We **should** then **find** that the relativistic Lorentz transformations reduce to the classical Galilean transformations as $v/c = \beta$ approaches zero. Applying such case into Lorentz transformations shows that as:

$$\beta \Rightarrow 0, \gamma \Rightarrow 1$$

All transformations reduce to the classical Galilean ones.

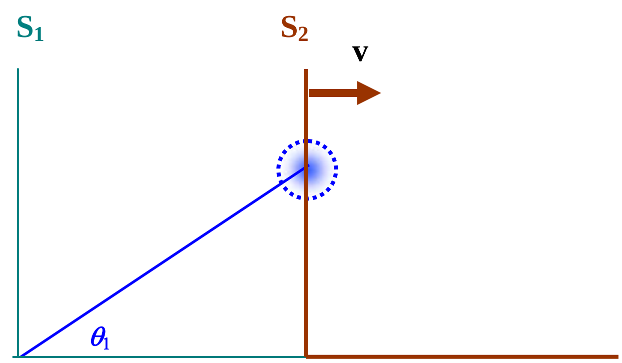
*Therefore, Lorentz transformations **do** indeed **agree** with the correspondence principle.*

The Doppler Effect

What is Doppler Effect?

It is the change in the ***measured frequency*** of a source, due to the motion of the source (and/or) the observer.

- *The Doppler effect* for sound waves travel in a medium, depends on **two** velocities: the source velocity and the observer velocity with respect to that medium.
- However, light and other electromagnetic waves require no medium. Therefore *the Doppler effect* for electro-magnetic waves depends on only **one** velocity: the relative velocity between the source and the observer.



Suppose that there is a source of frequency υ_o (and corresponding period T_o) at rest on the Y axis in S_2 .

And the source is moving with constant velocity *v* along the x-axis of the coordinate system S_1.

If we are at rest at the origin of S_1, we would measure the frequency of the source to be

$$\nu = \frac{\nu_0}{\gamma\left[1 + \beta\cos\theta_1\right]} = \nu_0 \frac{\sqrt{1 - v^2/c^2}}{1 + (v/c)\cos\theta_1} \qquad \text{......1}$$

In this relation, υ_o is measured in a coordinate system at rest with respect to the source (zero relative speed, thus the subscript zero). The observers measure the frequency υ, the relative speed v, and the angle θ_1 between the source position vector r_1 & + x-axis .

Some Special Cases

Equation 1 represents a general case, from which one may derive expressions for some other *special cases*.

Assume that you are at the origin of S_1, what the measured frequency of the source would be:

1- If the source of light were moving directly away from you:

In this case $\theta_1 = 0$, hence;

$$\nu = \nu_0 \frac{\sqrt{1 - v/c}}{\sqrt{1 + v/c}}$$

Note that the numerator is smaller than the denominator, giving *a lowered frequency* when the source is moving away from you, just as for SOUND waves. (However, the equation is different for sound waves.)

2- If the source of light were moving directly toward you:

In this case $\theta_1 = 180^\circ$, hence;

$$\boxed{\nu = \nu_0 \frac{\sqrt{1 + v / c}}{\sqrt{1 - v / c}}}$$

The numerator now is larger than the denominator, giving the expected *increase in frequency* when the source is moving toward you.

3- If the source were moving perpendicular to a line from you:

In this case (which known as **the transverse Doppler effect**) $\theta_1 = 90^\circ$, At this angle there is no relative motion toward or away from you, so the classical Doppler effect for mechanical waves would give $\upsilon = \upsilon_o$.

However, the same is not true for electromagnetic waves. With $\theta_1 = 90^\circ$, Eq. (1) becomes;

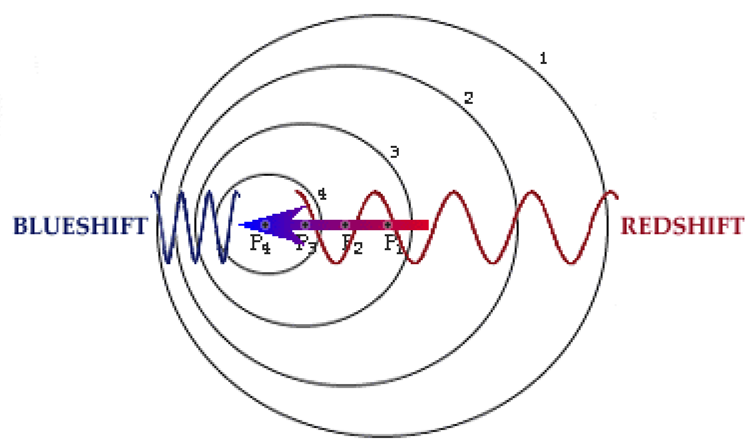
$$\boxed{\nu = \nu_0 \sqrt{1 - v^2 / c^2} = \nu_0 / \gamma}$$

The frequency decreases for this relativistic transverse Doppler effect.

So…

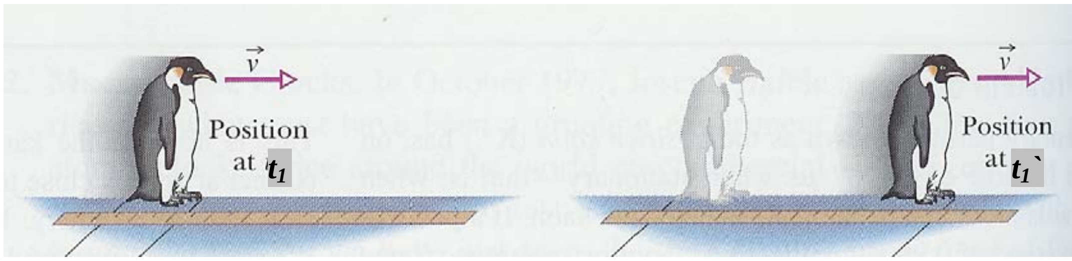
Doppler shifts in the frequencies of electromagnetic waves occur *not only for* **relative motion** toward or away from an observer, *but also for* **transverse motion**.

Astronomical Doppler effect



According to the Doppler Effect, the radiation emitted by an object moving toward an observer is squeezed; its frequency appears to increase and is therefore said to be **blueshifted**. In contrast, the radiation emitted by an object moving away is stretched or **redshifted**. Blueshifts and redshifts exhibited by stars, galaxies and gas clouds indicate their motions with respect to the observer.

Length Contraction



$x_1\,(t_1)$ $x_1`\,(t_1)$ $x_1\,(t_1)$ $x_1`\,(t_1`)$

If you want to measure the length of a penguin while it is moving, you must mark the positions of its front and back *simultaneously* (in your reference frame), as in (a), rather than *at different times*, as in (b).

Applying the Lorentz transformations to our two distances, we obtain;

$x_2 = \gamma\,(x_1 - vt_1)$ and $x_2` = \gamma\,(x_1` - vt_1)$

Subtracting, we obtain;

$$(x_2` - x_2) = \gamma\,(x_1` - x_1)$$

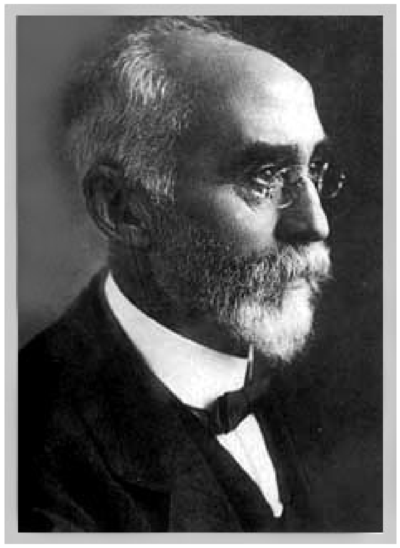
Note that $(x_2` - x_2)$ is the length as measured in S_2. Since the object is at rest with respect to S_2, let's call this length L_o. This gives us

$$\boxed{L = L_0\sqrt{1 - v^2/c^2} = \frac{L_0}{\gamma}}$$

Because the Lorentz factor γ is always greater than unity, then L is always less than L_o.

I.e. *The relative motion causes a length contraction.*

Because γ increases with speed v, the length contraction also increases with v.

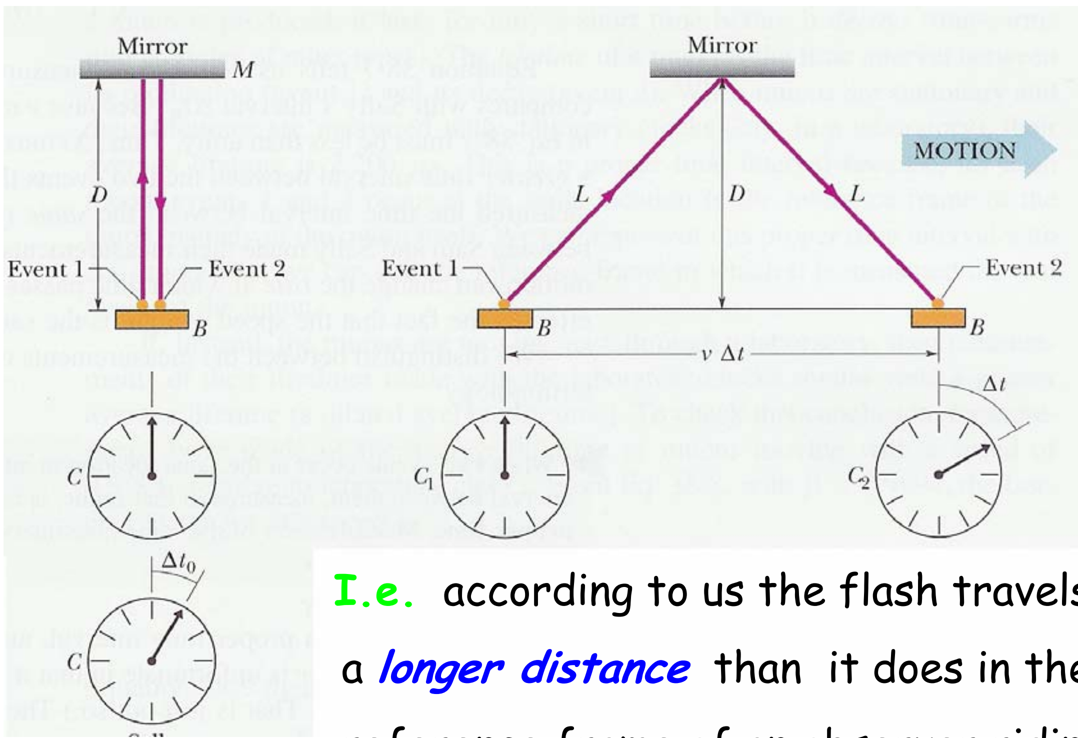
Remember: that $y_2 = y_1$ and $z_2 = z_1$.

Therefore, any lengths measured **perpendicular** to the direction of the motion **will not be changed** by the motion.

Length contraction occurs only along the direction of the relative motion.

Time Dilation

Suppose we travel inside a spaceship and watch a light clock. We will see the path of the light in simple **up-and-down motion**. If, instead, we stand at some relative rest position and observe the spaceship passing us by $0.5c$. Because the light flash keeps up with the horizontally moving light clock, we will see the flash following **a diagonal path.**



I.e. according to us the flash travels a ***longer distance*** than it does in the reference frame of an observer riding with the ship. Since the ***speed of light is the same*** in all reference frames (Einstein's second postulate), the flash must travel for a ***longer time*** between the mirrors in our frame than in the reference frame of an observer on board.

$$\Delta t_0 = \frac{2D}{c}$$

$$\Delta t = \frac{2L}{c}$$

$$L = \sqrt{\left(\tfrac{1}{2}v\Delta t\right)^2 + D^2}$$

$$\Rightarrow \quad \boxed{\Delta t = \frac{\Delta t_0}{\sqrt{1 - v^2/c^2}} = \gamma \Delta t_0}$$

This stretching out of time is called **time dilation**.

Some numerical values:

* Assume that $v = 0.5c$,then $\gamma = 1.15$, so $T = 1.15\ T_0$ .This means that if we viewed a clock on a spaceship traveling at half the speed of light, we would see the second hand take 1.15 minutes to make a revolution, whereas if the spaceship were at rest, we would see it take 1 minute.

* If the spaceship passes us at 87% the speed of light, $\gamma = 2$; and $T = 2\ T_0$. We would measure time events on the spaceship taking **twice the usual intervals**. **i.e.** the hands of a clock on the ship would turn only half as fast as those on our own clock.

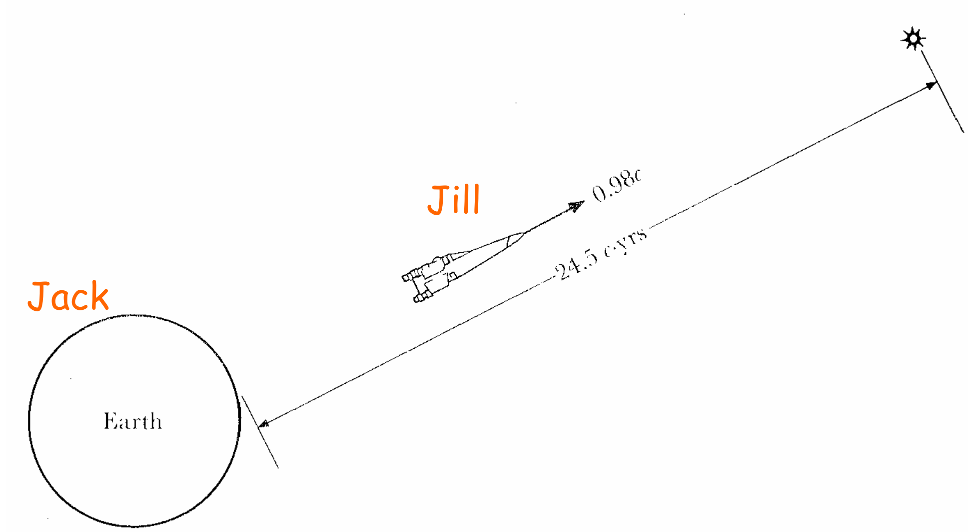
* If it were possible to make a **clock fly** by us at **the speed of light**, **the clock would not appear to be running at all**. We would measure the interval between ticks to be infinite.

* ***Time dilation*** has been confirmed in the laboratory countless times with particle accelerators. The lifetimes of fast-moving radioactive particles increase as the speed goes up, and the amount of increase is just what Einstein's equation predicts.

Example 4.6: (The Twin Paradox)

Jack and Jill are 25-year-old twins. Jack must stay on earth, but the astronaut Jill travels at **0. 98c** to a star **24.5** light years away and returns immediately. Ignoring the end-point acceleration times, find the twins' ages when she returns.

(One light year = 1 *C* . yr, the distance light travels in one year.)



Solution:

From Jack earth-bound frame of reference;

Jill travels a total distance of 49 light years (out and back) at 0.98c. Thus; the total time of her journey as *Jack* measure it is;

T_{Jack}= 49 c . yrs / 0.98c. = 50 years

Therefore 50 years of earth time have passed, so Jack is (25 + 50) years = 75 years old. However, this 50 years is dilated time for Jill's frame of reference.

Since γ = **5** for **v = 0.98c,**

T_{Jill} = 50 years / 5 = 10 years

Jill therefore is (25 + 10) years = 35 years old. She is 40 years younger than her brother.

Question??

Since the choice of frame of reference is relative, **why don't we place Jill in S1?** She then sees the earth move away and return, and therefore it is Jack who has travelled out and back at 0.98c. He should be the one who is 40 years younger.

Since they both can't be 40 years younger, this apparent contradiction is called *the twin paradox*.

Answer:

❶ Recall that we are dealing with **the special theory of relativity, which refers to inertial reference frames.** In the twin paradox, the earth is an approximately inertial reference frame, but Jill's spaceship isn't. The choice of frames of reference is relative in the special theory of relativity *only if the frames of reference are all inertial*. Therefore an attempt to use the special theory in a no inertial frame of reference causes incorrect results. So, **Jack does age more rapidly than Jill.**

❷ Experiments (such as the clocks in jetliners) confirm this prediction.

❸ **Length contraction** can be used, as well, to solve this problem:

According to Jill spaceship frame of reference;

Jill travels (out and back)a total distance of:

$L_{Jill} = L_{Jack} / \gamma = 49$ c . yrs /5 = 9.8 c. yrs

Since she travels at 0.98c. Thus; the total time of her journey as *she* measure it is;

$T_{Jill} = 9.8$ c . yrs / 0.98c. = 10 years

Which confirm the previous prediction.

The Relativity of Velocities

Suppose we wish to use the Lorentz transformation equations to compare the velocities that two observers in different inertial reference frames S_1 and S_2 would measure for the same moving particle.

Let S_2 moves with velocity v relative to S_1, and there is a particle in S_1 moving with constant velocity v_{1x} parallel to the x-axis. If the particle position is (x_1, y_1, z_1) at the instant t_1 , then using *Lorentz coordinate transformations* :

$$x_2 = \gamma(x_1 - vt_1) \;,\; y_2 = y_1 \,, \quad z_2 = z_1 \quad \text{and} \quad t_2 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right)$$

Differentials of those equations are;

$$dx_2 = \gamma(dx_1 - vdt_1),$$

$$dy_2 = dy_1 \,, \quad dz_2 = dz_1 \qquad \text{and}$$

$$dt_2 = \gamma\left(dt_1 - \frac{v}{c^2}dx_1\right)$$

Dividing dx_2 , dy_2 and dz_2 by dt_2 to obtain the velocity components gives;

$$v_{2x} = \frac{v_{1x} - v}{1 - v_{1x}.v / c^2}$$

$$v_{2y} = \frac{v_{1y}}{\gamma[1 - v_{1x}.v / c^2]}$$

$$v_{2z} = \frac{v_{1z}}{\gamma[1 - v_{1x}.v / c^2]}$$

These formulas are known as ***Lorntz velocity transformations.***

Note:

1- Both v_{2y} and v_{2z} depend on v_{1x}, because the time t_2 depends on the x position of the particle.

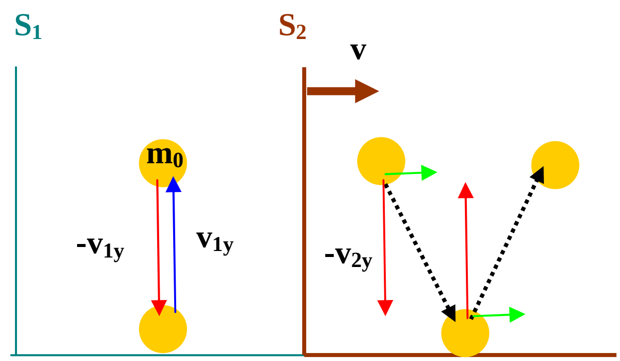
2- At small v ($\beta \Rightarrow 0$), and the relativistic velocity transformations will reduce to the classical, or Galilean, velocity transformations.

3- No *Lorntz velocity transformations* can give a speed greater than *c.*

Relativistic Mass & Momentum

- In classical physics when two bodies collide together, the total *mass, energy and momentum* before and after the collision are equal.
- Let us apply conservation laws to viewers from two different inertial reference frames S_1 and S_2.

If someone in S_1 throws a ball with mass m_0 to make an elastic collision with the ground,then the *conservation law of momentum* in his frame requires that:



$$\Delta P_{1y} = 2m_0 v_{1y}$$

For an observer in S_2 the *conservation law of momentum will* requires that*:*

$$\Delta P_{2y} = 2mv_{2y} = 2mv_{1y}\sqrt{1 - v^2 / c^2}$$

(recall: v_{1x}=0)

But **the principle of relativity** demands that *the laws of physics are the same in all inertial reference frames*. Hence;

$$\Delta P_{1y} = \Delta P_{2y}$$

$$m_0 = m\sqrt{1 - v^2 / c^2}$$

or;

$$\boxed{m = m_0 / \sqrt{1 - v^2 / c^2} = \gamma\, m_0}$$

This is the relativistic mass transformation.

Notes:

1- If $v \ll c$ then m is effectively equal to the rest mass m_0 (*the classical limit*). When we refer to the mass of an electron as 9.1×10^{-31} kg we mean its rest mass.

2- As the velocity of a body increases the relativistic mass becomes significantly greater than the rest mass. The relativistic mass of a body traveling at about 0.99c is roughly seven times its rest mass.

3- As **v** $\longrightarrow$ **c** , ***the mass*** $\longrightarrow$ ***infinity***. This huge increase in **inertia** makes it impossible to accelerate bodies of non-zero rest mass up to the velocity of light.

4- Since ***p = mv*** , the relativistic linear momentum can be written as;

$$\boxed{p = m_0 v / \sqrt{1 - v^2 / c^2} = \gamma\, m_0 v}$$

Relativistic Force

We can use Newton's second law to define ***force*** by the relation F = dp/dt. So we have;

$$\boldsymbol{F} = \frac{d}{dt} m v = m\boldsymbol{a} + \boldsymbol{v}\frac{dm}{dt}$$

If the force is perpendicular to the velocity, the force can't do any work on the particle, so the speed won't change. This happens in ***uniform circular motion***. The direction of v changes, but the magnitude of v doesn't. Therefore m doesn't change and dm/dt=0. Substituting for $m = \gamma m_0$, we have

$$\boxed{F_{\perp} = \gamma m_0 a}$$

However, if the force is parallel to the velocity, the particle speed and mass will change. Then;

$$\frac{dm}{dt} = \frac{d}{dt}\frac{m_0}{\left(1 - v^2 / c^2\right)^{1/2}}$$

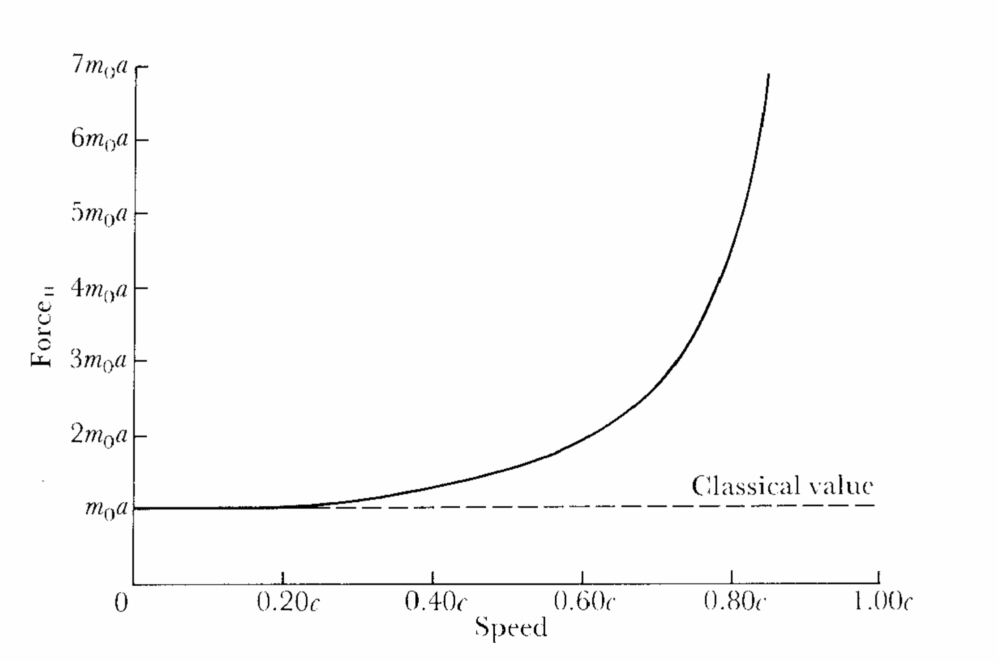
and;

$$\boxed{F_{\parallel} = \gamma^3 m_0 a}$$

Notes:

1- $\boxed{F_{\parallel}}$ is much larger than $\boxed{F_{\perp}}$.

2- $\boxed{F_{\parallel}}$ increases rapidly as v gets close to c.



Fact:

To approach c we need an infinite force to accelerate an infinite mass.

Relativistic Energy

Work may be done on a body to increase its kinetic energy, *KE*.

What is the expression of *KE* in relativistic physics?

Let's start an object from rest with a net external force F in the (+ve) x direction. Then the work done by F will be stored in the form of kinetic energy.

That is,

$$KE = \int F dx = \int m_0 \left(1 - v^2 / c^2\right)^{-3/2} v dv$$

which integrates to

$$KE = m_0 c^2 \left(1 - v^2 / c^2\right)^{-1/2} - m_0 c^2 = \gamma m_0 c^2 - m_0 c^2$$

or most simply;

$$\boxed{KE = mc^2 - m_0 c^2 = \Delta mc^2} \qquad (1)$$

where $\Delta m = m - m_o$ is *the relativistic mass increase*.

Suppose a body with a rest energy of E_0 undergoes a work that increases its KE, then its total energy will be $E = E_0 + KE$, or;

$$\boxed{KE = E - E_0} \qquad (2)$$

Comparing (1) & (2), we can say that:

$$\boxed{E = mc^2} \quad \text{and} \quad \boxed{E_0 = m_0 c^2} \qquad (3)$$

This famous equation states *the equivalence of mass and energy*. Therefore, anything that has a mass m has an energy $E = mc^2$, and anything that has an energy E has a mass $m = E/C^2$. That is;

Energy and mass are just two equivalent ways of describing the same thing.

Using the relation of *the relativistic mass* along with (3), we obtain

$$\boxed{E = E_0 / \sqrt{1 - v^2/c^2} = \gamma E_0} \qquad (4)$$

Equation (4) as other *relativistic* expressions shows that for objects with a nonzero rest mass, ***c* is the upper limiting speed**. This doesn't forbid the existence of particles that have zero rest mass and which can only move at $v = c$.

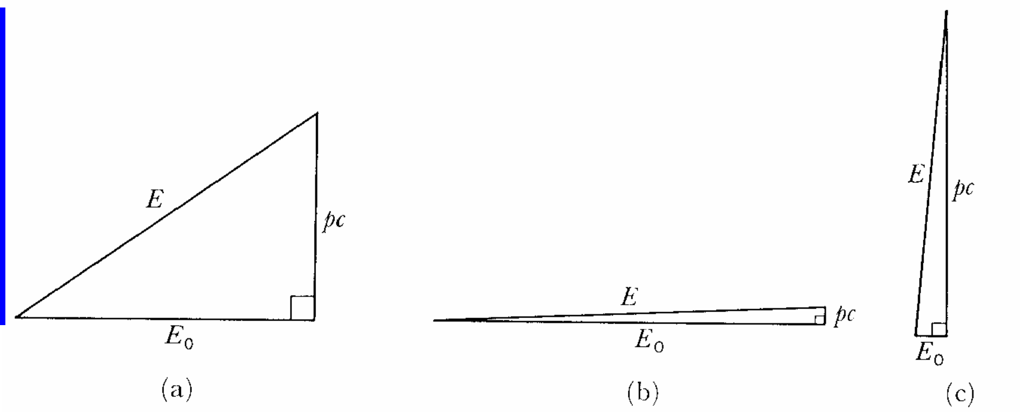
Again starting with $m = mo/(1 - v^2/c^2)^{1/2}$, if we square both sides and rearrange terms recognizing mv as the magnitude of the linear momentum p, we get;

$$\boxed{E^2 = p^2c^2 + E_0^2} \qquad (5)$$

a) Eq. (5)

b) when v<<c

c) when $v \to c$



The unit conversions:

Since it's easier to determine the work done and thus KE in **electron volts** for charged atomic and subatomic particles, **eV-** based units are most often used as following;

1- When a particle is described as a 1-MeV particle. This means that the kinetic energy of this particle is 1-MeV.

2- According to ($E/c^2=m$) we can use the units of **eV/c^2, KeV/c^2, MeV/c^2,ect** to specify the mass units.

3- According to (5) we can use the units of **eV/c, KeV/c, MeV/c,ect** to specify the momentum units.

General Theory of Relativity

Introduction:

* In introductory physics we learn that:

- The acceleration of an object of mass m is inversely proportional to m; $a = F_{net}\dfrac{1}{m_i}$ (Newton's 2nd law)

- The gravitational force acting on an object of mass m is proportional to m; $F_{grav} = 9.8\left(m/s^2\right)\times m_g$ (Newton's universal law of gravitation)

* On the face of it, these two properties of the mass m are totally different. I.e. there is no a clear link between the property governing how hard gravity pulls on the object (m_g) and the property governing its reluctance to accelerate (m_i).

* Although this link was hidden into Galileo old statement: "*in the absence of friction, all falling bodies accelerate at the same rate*", no one could read it correctly and get used of it.

* Albert Einstein was the first to discover the surprising results that arise by postulating that m_g and m_i are the same.

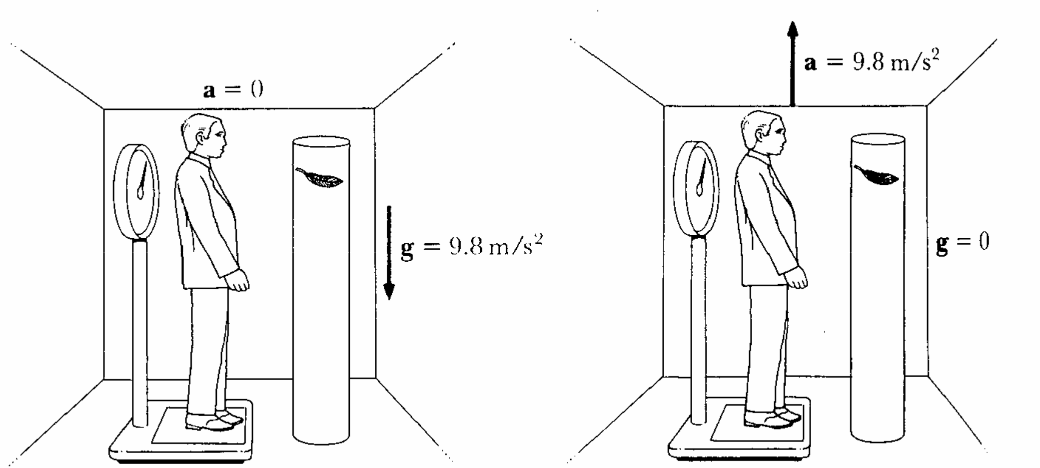
Principle of Equivalence:

General relativity is a **theory of gravitation** developed between the years **1907–1915**. It began with *the principle of equivalence* introduced by Einstein in **1907**.

* Suppose that you are in a small closed room. You drop an object in a vacuum from rest and you find that its acceleration toward the floor is **9.8** m/s2. Where is your room?

One possible answer: your room is at rest on the surface of the earth, where all freely falling bodies accelerate downward at **g = 9.8** m/s2.

Another possible answer: your room is in outer space (in a region where **g=0**), but your room is accelerating "upward" at a constant 9.8 m/s2.



(a) On earth (b) In outer space

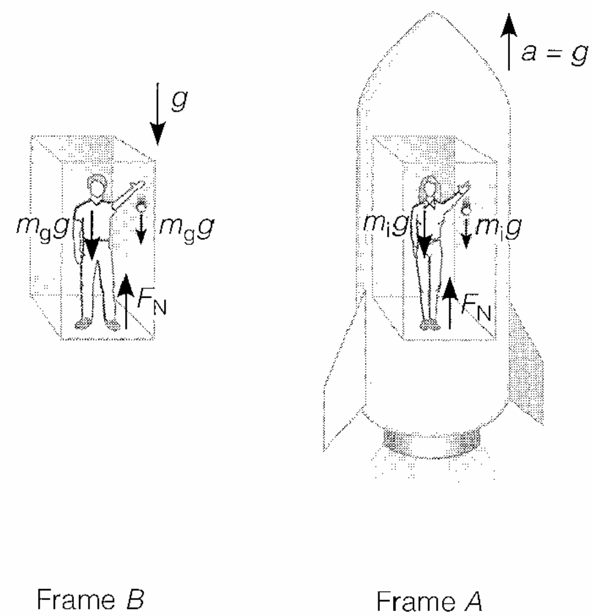
* If *special relativity* begins with Einstein's simple postulate that the speed of light is the same in all frames.

General relativity begins with another of Einstein's "simple" postulates which is:

Inertial mass m_i and gravitational mass m_g are the same.

* In particular, **if $m_g = m_i$**, it would be impossible to determine whether we are in an inertial frame passed through by a uniform gravitational field, or in a frame which accelerates at a constant rate but there is no field.

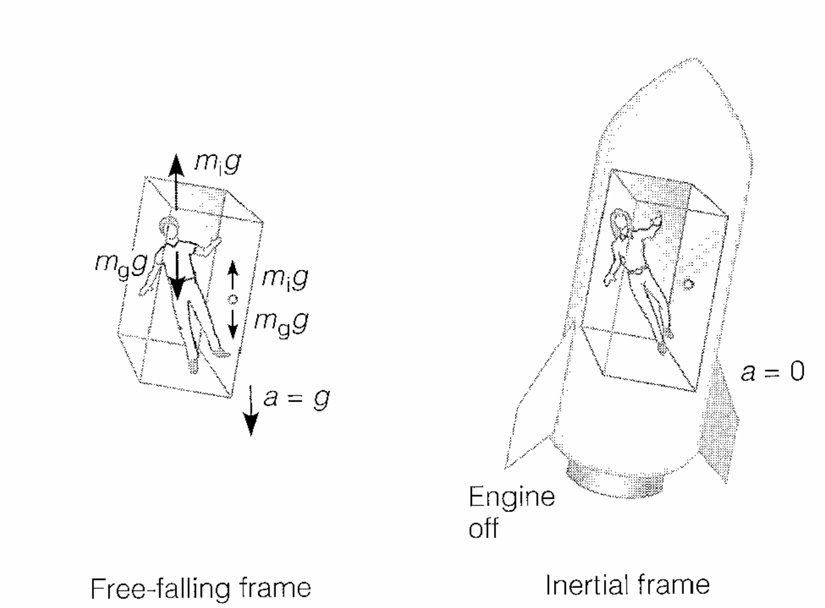
Also, no mechanical experiment would be able to distinguish a frame that is accelerating in "free-fall" in a uniform gravitational field from one that is inertial and without a gravitational field.

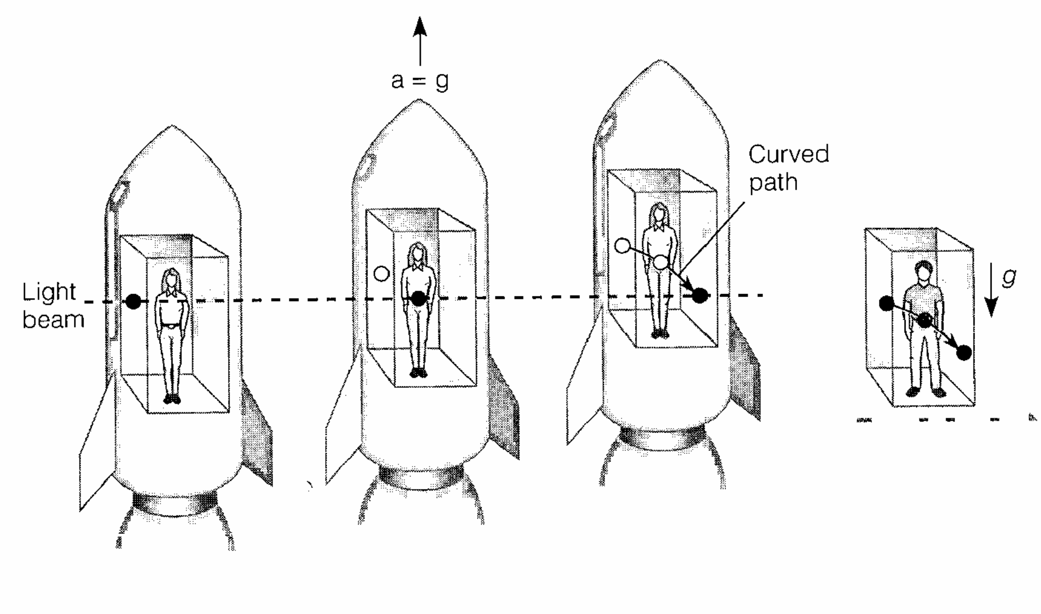


An observer in the inertial frame would see all objects floating or moving at constant velocity, because no forces act.

An observer in the free-falling frame would also see objects seemingly moving at constant velocity because the *gravitational force* ($m_g g$) is exactly canceled by the *inertial force* ($-m_i a$).

Similarly, *no optical experiment* would be able to distinguish between a *uniform gravitational field*, and a frame which *accelerates at a constant* rate but there is no field





(a) Accelerating frame

(b) Equivalent frame

The point is that *there is no way of deciding from your experiment* whether your closed room is *at rest in the earth's gravitational field* or *accelerating in distant space*.

This is the basis of Einstein's ***principle of equivalence***:

No experiment can distinguish between a uniform gravitational field and an equivalent uniform acceleration.

This means:

- Free fall (*acceleration inside a gravitational field*) is an inertial motion.
- Inertial mass mi and gravitational mass mg are the same.
- If all accelerated systems are equivalent, then **Euclidean geometry** cannot hold in all of them. Thus the ***equivalence principle*** led Einstein to search for a gravitational theory which involves **curved space-time**.

Space-Time in the General Relativity

❑ The Curvature of Spacetime:

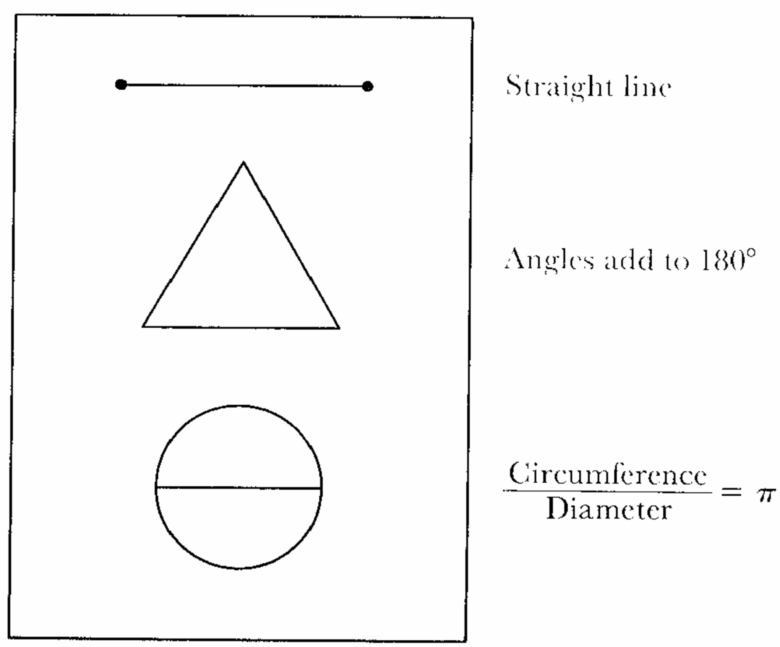
* **Recall:** The equations of special relativity follow the rules of a **flat spacetime**, (i.e. Euclidean geometry).

* If all accelerated systems are equivalent, then Euclidean geometry cannot hold in all of them. In fact, the *equivalence principle* led Einstein to search for a gravitational theory which involves **curved spacetime**.

But, what do we mean by a curved spacetime?

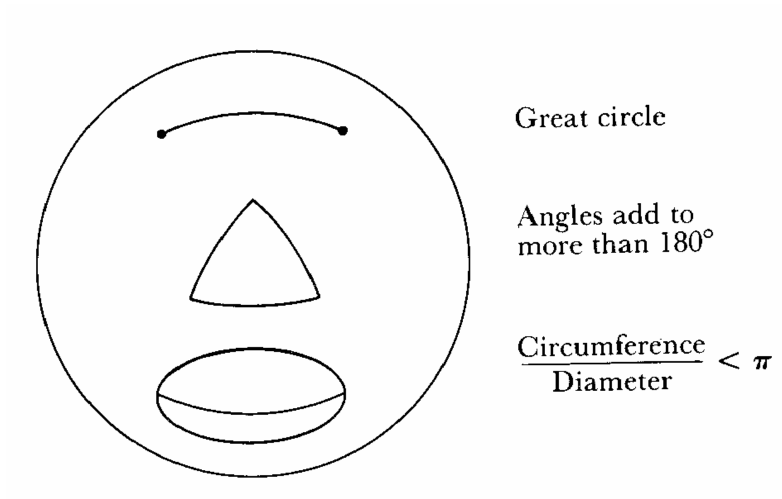
* A **Flat spacetime** can be recognized by some basic Euclidean rules such as:

The shortest distance between two points is a straight line, parallel lines can not meet at any point, the angles of a triangle add to 180°, and the ratio of the

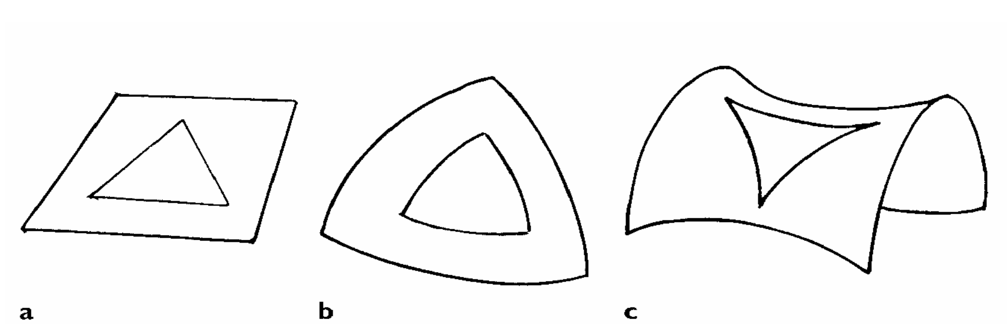


circumference to diameter of a circle is equal to π.

* The rules of Euclidean geometry are only valid in flat spacetime, but if we draw these figures on a curved surface like a sphere object, the Euclidean rules no longer hold. In this case, a curved line is the shortest distance between two points, lines parallel at the equator meet at the poles, the angles of a triangle add to more than 180°, and the ratio of the circumference to diameter of a circle is less than π.



* If we draw these figures on a curved surface like a saddle-shaped object, different results will be obtained.



* In general, if we measure the sum of the angles for a triangle in a space-time we call that space:

(a) Flat if the sum is **equal to 180°**.

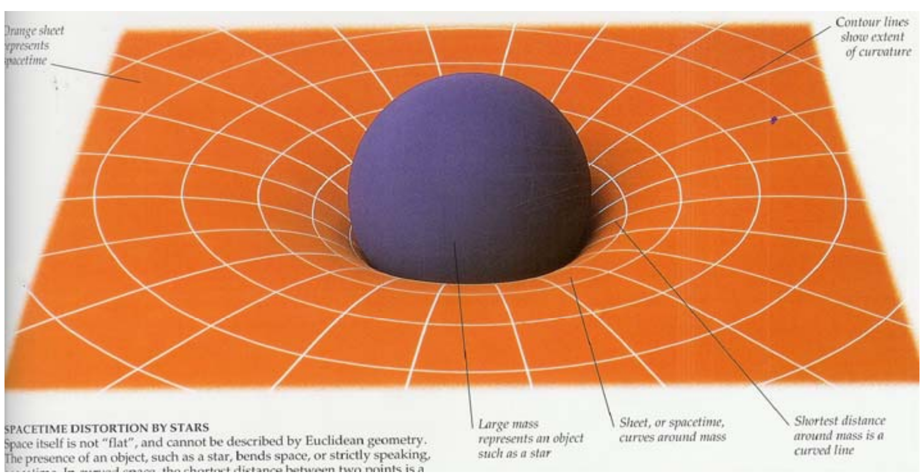
(b) **Spherelike** or **positively curved** if it is **larger than 180°**.

(c) **Saddlelike** or **negatively curved** if it is **less than 180°**.

❑ Our Universe & Curved Spacetime:

In general relativity massive objects do not directly impart a force on other massive objects as hypothesized in Newton's action at a distance idea. Instead objects respond to how the massive object curves space-time.

The curvature of space-time can be viewed in the following way. Placing a heavy object such as a bowling ball on a rubber mat will produce a ' cavity ' in that mat.



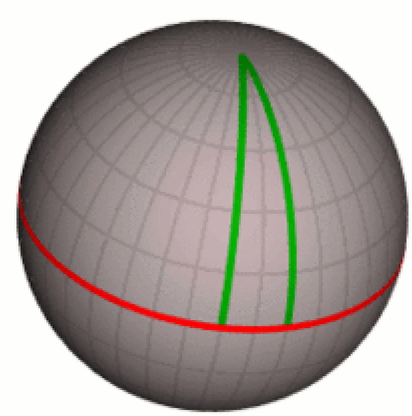
This is analogous to a large mass such as the Earth causing the local space-time geometry to curve. **The larger the mass, the bigger the amount of curvature.** A relatively light object, such as a ping-pong ball, placed in the vicinity of the ' cavity ', will accelerate towards the bowling ball in a manner governed by the ' cavity '. This is analogous to the Moon orbiting the Earth.

❑ The New Laws of Curved Spacetime:

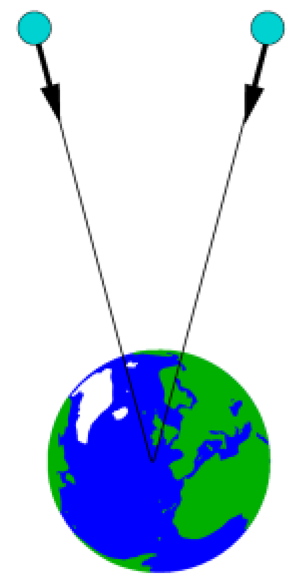
Now we need to know **first** the rules for the behavior of a particle in a curved spacetime, and **second** the rules to determine the curvature itself.

1- Motion of a particle & (Geodesics):

The "straightest" or shortest distances between two points on the curved surface are called ***geodesic lines*** or simply ***geodesics***. Objects that were initially traveling in parallel paths through flat space-time come to travel in a non-parallel fashion through curved spacetime.



This effect is called geodesic deviation, and it is used in general relativity as an alternative to gravity.



For example, two balls initially at rest with respect to and above the surface of the Earth come to have a converging component of relative velocity as both accelerate towards the center of the Earth due to their free-fall.

2-The shape of spacetime & Einstein field equations (EFE):

The Einstein field equations (EFE) describe how stress-energy causes curvature of spacetime and are usually written in tensor form:

$$\boxed{R_{ij} - \tfrac{1}{2} g_{ij} R = \kappa T_{ij}}$$

where R_{ij} is the curvature tensor, R the scalar curvature, g_{ij} the *metric* tensor and T_{ij} the stress-energy tensor. The constant κ is called the *Einstein constant of gravitation,* which equals to

$$\kappa = -\frac{8\pi G}{c^4}$$

where G & c are the universal gravitational constant and the speed of light respectively. One can write the EFE in a more compact form by defining the *Einstein tensor*

$$G_{ij} = R_{ij} - \tfrac{1}{2} g_{ij} R$$

Then, EFE can then be written as

$$\boxed{G_{ij} = -\frac{8\pi G}{c^4} T_{ij}}$$

* Each subscript i or j stands for one of the 4 coordinates of space-time; and have the range 1 to 4. Therefore, what looks like one equation is actually 4 x 4 = 16 equations. But since some are repeated there are really 10 equations.

* EFE relates the **curvature** of space-time (G_{ij}) to the **stress-energy** (T_{ij}), which is the source of the gravitational field. The tensor T_{ij} includes stress, momentum, and energy (**i.e**, it includes mass the source for Newtonian gravity).

* A brief description of the field equations was given by the physicist *John Wheeler;*

Mass tells spacetime how to curve, and spacetime tells mass how to move.

❑ Solutions of Einstein's equations & Metrics:

Indeed, EFE are amongst the most difficult equations in science. However, some exact solutions have been found to get use of them.

* Each solution of Einstein's equations is called a ***metric***, and it is *a formula that allows us to compute the spacetime interval between any two nearby points* (**i.e, geodesic**).
If we know the, metric at every point, we know everything about the space, including its curvature.

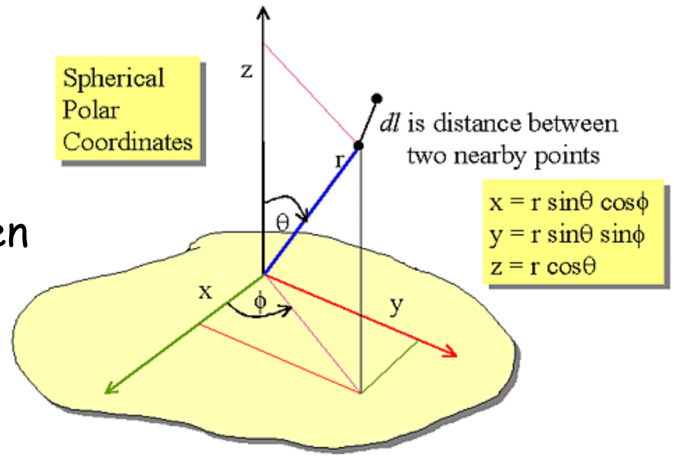
* In **rectangular coordinates**, the distance between the points (x, y, z) and $(x+dx, y+dy, z+dz)$ is given by

$$ds^2 = dx^2 + dy^2 + dz^2$$

The set of coefficients of ds^2 can be written in the form of a matrix. These numbers define the **metric** (g)for rectangular coordinates in three dimensions;

$$g = \begin{bmatrix} g_{xx} & g_{xy} & g_{xz} \\ g_{yx} & g_{yy} & g_{yz} \\ g_{zx} & g_{zy} & g_{zz} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

*The form of the metric for a given space is not unique, in **polar coordinates**, for example, the distance between the points (r, θ, ϕ) and ($r+dr$, $\theta+d\theta$, $\phi+d\phi$) is



$$dl^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2$$

And hence the **metric** is;

$$g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2 \sin^2\theta \end{bmatrix}$$

* The form of the metric determines the properties of the space. The space-time of relativity is a four-dimensional vector space in which the " *points* " are "***events*** " .

* The distance between the *events* whose coordinates are (x, y, z, ct) and ($x+dx$, $y+dy$, $z+dz$, $ct+cdt$) is given by

$$ds^2 = cdt^2 - dx^2 - dy^2 - dz^2$$

Because of those *minus signs*, the space-time of *special relativity* is not quite Euclidian; it is sometimes called **pseudo-Euclidian**.

* In **4D-spacetime** the ***metric*** is a symmetric 16 component tensor as ;

$$g_{ij} = \begin{bmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{bmatrix}$$

But since some are repeated there are really 10 equations;

$$\begin{aligned} ds^2 = g_{11}dx_1^2 + g_{22}dx_2^2 + g_{33}dx_3^2 + g_{44}dx_4^2 + 2g_{12}dx_1\,dx_2 + 2g_{13}dx_1\,dx_3 \\ + 2g_{14}dx_1\,dx_4 + 2g_{23}dx_2dx_3 + 2g_{24}dx_2dx_4 + 2g_{34}dx_3dx_4 \end{aligned}$$

❑The Schwarzschild metric:

* A German scientist **Karl Schwarzschild** found the first exact solution of Einstein's equations in **1916**, just months after the publication of these equations.

The **metric** found by Schwarzschild is

$$\boxed{ds^2 = \left(1 - \tfrac{r_s}{r}\right)c^2 dt^2 - \frac{dr^2}{1 - \frac{r_s}{r}} - r^2 d\theta^2 - r^2 \sin^2\theta \, d\phi^2}$$

Where r_s is the Schwarzschild radius (in meters) of the massive body, which is related to its mass M by;

$$\boxed{r_s = \frac{2GM}{c^2}}$$

where G is the gravitational constant.

* This metric describes the *gravitational field* outside a spherical, non-rotating mass such as a (non-rotating) star, planet, or black hole. It is also a good approximation to the gravitational field of a slowly rotating body like the Earth or Sun.

* The Schwarzschild solution appears to have singularities at $r = 0$ and $r = r_s$; such solutions are now believed to exist and are termed *black holes*.

❑ Fundamental Principles:

* General relativity states that the laws of physics must be the same for all observers (accelerated or not).

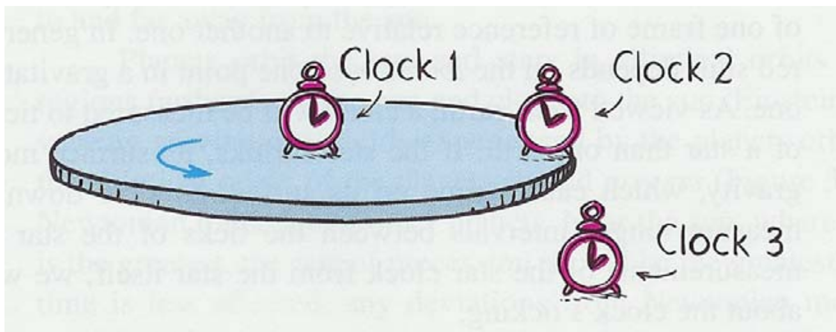
* General relativity is a *metric* theory of gravitation. The main defining feature is **the concept of gravitational 'force' being replaced by spacetime geometry**. Phenomena that in classical mechanics are ascribed to the action of the force of gravity (such as free-fall, orbital motion, and spacecraft trajectories) are taken in general relativity to represent inertial motion within a curved geometry of spacetime.

* Finally, the curvature of spacetime and its energy-momentum content are related, this relationship is determined by the Einstein field equations.

Some of General Relativity Predictions

(1) Gravitational red-shift:

Suppose we measure time with three identical clocks, one placed on the center of a rotating disk, a second placed on the rim of the disk, and the third at rest on the ground.



* From the laws of special relativity we know that the clock attached to the center, since it is **not moving** with respect to the ground, should run at the same rate as the clock on the ground-but not at the same rate as the clock attached to the rim of the disk.

* An observer on the rotating disk and an observer at rest on the ground both see the clock on the rim run more slowly than their own clocks. However, explanations of the difference for the two observers are not the same.

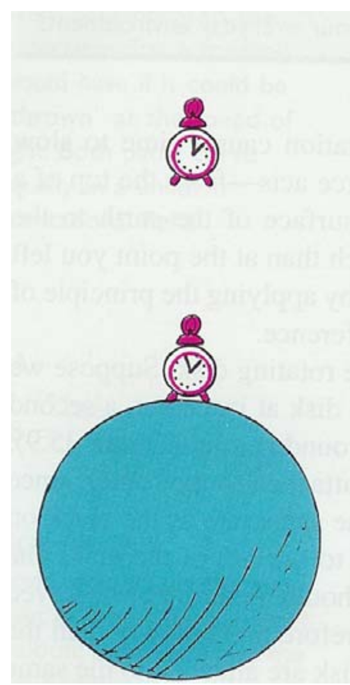
* To the observer on the ground, the slower rate of the clock on the rim is due to its motion. The observer on the disk is likely to conclude that the *centrifugal force* has something to do with the slowing of time. He notices that as he moves in the direction of the centrifugal force, outward from the center to the edge of the disk, time is slowed.

* By applying ***the principle of equivalence***, which says that *any effect of acceleration can be duplicated by gravity*, we must conclude that:

As we move in the direction that a gravitational force acts, time will slow down.

* An executive working on the ground floor of a tall city skyscraper will age more slowly than her twin sister working on the top floor. The difference is very very small.

* For larger differences in gravitation, like between the surface of the sun and the surface of the earth, the differences in time are larger.



A clock at the surface of the sun should run measurably slower than a clock at the surface of the earth.

* Similarly, light emitted at one point in a gravitational field will have a different frequency if observed at a different point.

I.e. an atom on the sun should emit light of a lower frequency (slower vibration) than light emitted by the same kind of atom on the earth. This effect is called ***the gravitational red shift***.

* The gravitational red shift is observed in light from the sun, but some disturbing influences prevent accurate measurements of this tiny effect. It wasn't until 1960 when the gravitational slowing of time was confirmed using gamma rays from radioactive atoms.

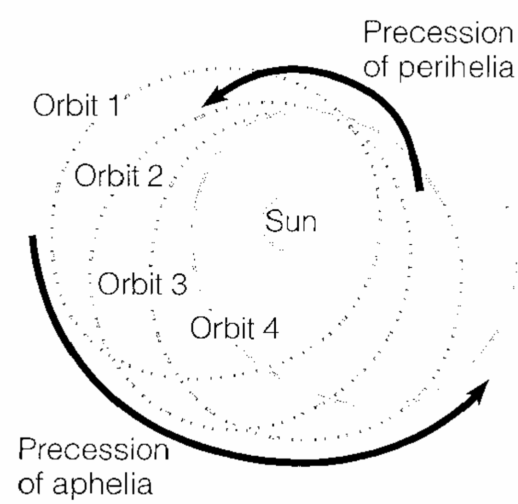
*** Conclusion:**

Measurements of time depend not only on relative motion, as we learned in special relativity, but also on the gravity (i.e. on the location of one point in a gravitational field relative to the other one).

(2) The precession of the perihelion of Mercury:

* The orbits of the planets about the Sun are not exactly circular, but slightly elliptical. At one point in its orbit, called *aphelion*, a planet will be slightly farther than average from the Sun, and at another, called *perihelion*, slightly closer.

* As long as a system is simply one object orbiting another, it is a direct prediction of classical Newtonian gravitation that the same path in space is repeated for ever.



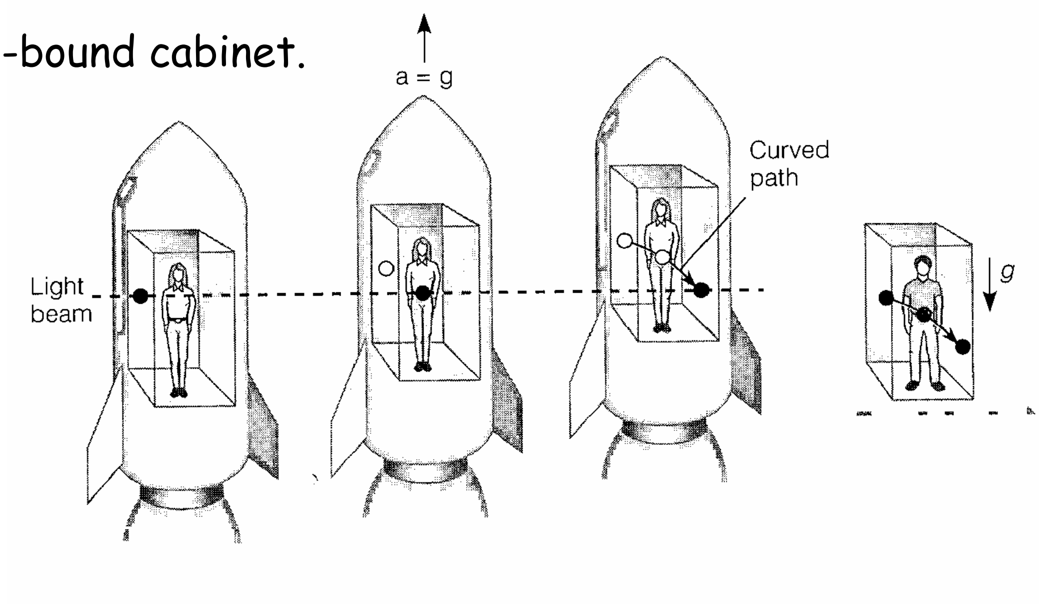
* But if anything interferes with that simple interaction, **the orbit will precess**, (i.e. the points of aphelion and perihelion gradually creep around in a circular fashion).

* Newtonian measurements of the rate of ***Mercury's precession*** did not agree exactly with observation. There is still 43 seconds of arc per century missing.

* Using general relativity, a correction to the classically expected precession rate of Mercury can be calculated. The result of 43 s of arc per century is in good agreement with observation.

(3) The deflection of light by the sun:

* Since a laterally moving light beam would appear to curve toward the floor in a rocket-powered cabinet, applying ***the principle of equivalence,*** it must curve toward the floor in an Earth-bound cabinet.



(a) Accelerating frame

(b) Equivalent frame

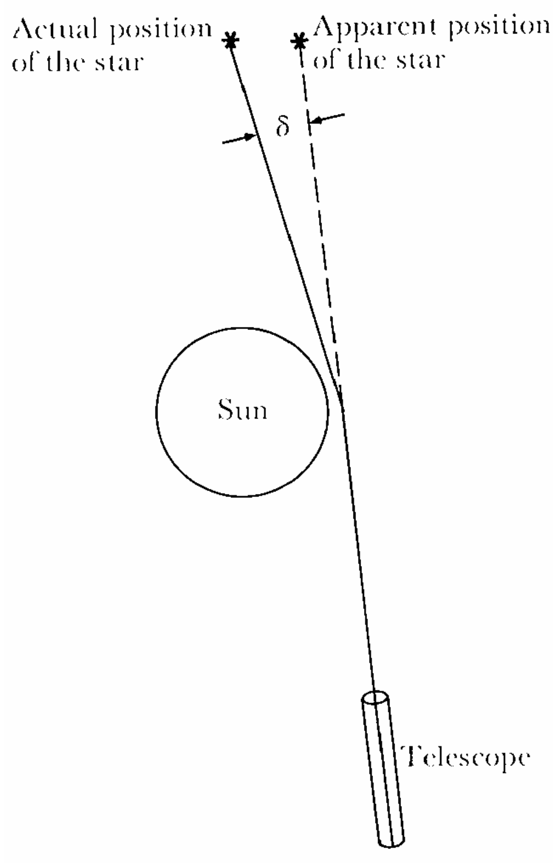
* In other words, if light is deflected by **acceleration**, it must be deflected by **gravity**.

But how can gravity bend light which is massless?

Einstein's first answer was that:" Gravity pulls on the energy of light because energy is equivalent to mass".

Later he gave a deeper explanation, that light bends when it travels in a space-time geometry that is bent (as we shall see later).

* Einstein predicted that starlight passing close to the sun would be deflected by an angle of 1.75 s of arc, which is large enough to be measured.



* Although stars are not visible when the sun is in the sky, the deflection of starlight can be observed during *an eclipse* of the sun. A photograph taken of the darkened sky around the eclipsed sun reveals the presence

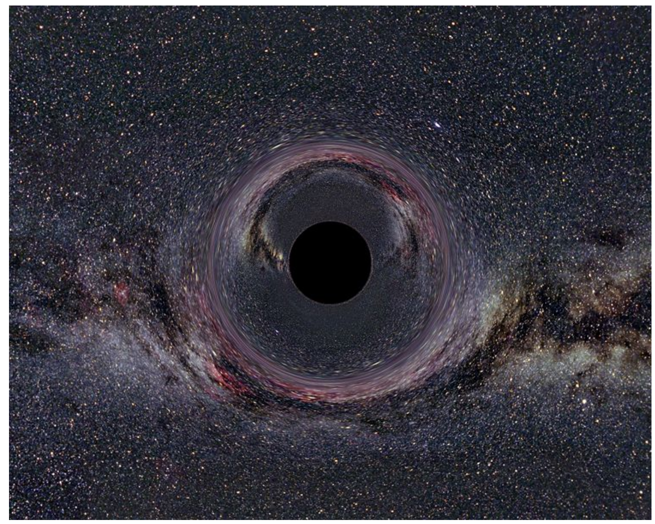
of the nearby bright stars. The positions of the stars are compared with those in other photographs of the same area taken at other times in the night with the same telescope. In every instance, the deflection of starlight has supported Einstein's prediction.

* On the other hand, it is theoretically possible for an object to be so dense that light simply cannot escape its gravitational potential energy at all. Such objects are known as ***black holes***.

BLACK HOLES

❑What is a Black Hole?

* A black hole is a region of spacetime that has so much mass concentrated in it that nothing can escape its gravitational pull, even light.



* The name 'black hole' was invented by *John Wheeler* in **1967.** Before that it was called 'Frozen Star'.

* When a large star has burnt all its fuel it explodes into a *supernova*. The stuff that is left collapses down to an extremely dense object known as **a neutron star.** If the neutron star is too large, the gravitational forces overcome the pressure gradients and collapse cannot be stopped. The neutron star continues to shrink until it finally becomes a black hole.

❑ Escape Velocity & Schwarzschild* Radius:

The speed with which you need to throw a rock in order that it just escapes the Earth's gravity is called the "**escape velocity**."

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* **Pronounced " SHVARTZshilt"**

The escape velocity from the surface of a star depends upon the size and mass of the star. The Newtonian expression for *escape velocity* is:

$$v_{esc} = \sqrt{\frac{2GM}{R}}$$

where **R** is the radius and **M** the mass of the star. If light is retarded by gravity then it will just fail to escape from the surface **when the escape velocity equals the speed of light**.

$$c = \sqrt{\frac{2GM}{R}}$$

This can be rearranged to give expressions for ***the radius inside which a particular mass must be compressed to form a black hole***, as;

$$\boxed{R_s = \frac{2GM}{c^2}}$$

$\mathbf{R_s}$ is the **Schwarzschild radius,** it is used to represent this radius since Newtonian theory gives the same result as the general relativistic analysis carried out by **Schwarzschild in 1916.**

Hence, **the critical density** ρ_c of spherically distributed matter **that will collapse to form a black hole**, is;

$$\rho_c = \frac{3M}{4\pi R_s^3} = \frac{3c^6}{32\,G^3 M^2 \pi}$$

The significant point about the density equation is that **critical density is inversely proportional to the square of the mass of the collapsing object**. This means that, whilst objects of relatively low mass like the Earth or Sun would need to reach incredibly high densities to form black holes, much larger bodies like galaxies or clusters have a much lower critical density.

	The Earth	**A galaxy**
Mass (kg)	6×10^{24}	10^{42}
Radius (m)	6.4×10^{6}	10^{21}
ρ_c ($kg.m^{-3}$)	2×10^{30}	7×10^{-5}
R_s (m)	8.9×10^{-3}	1.5×10^{15}
Remarks	The Earth must be compressed into a sphere of radius less than 1 cm to be a black hole.	ρ_c is less than the density of air by 4 orders of magnitude.

❑ Event horizon:

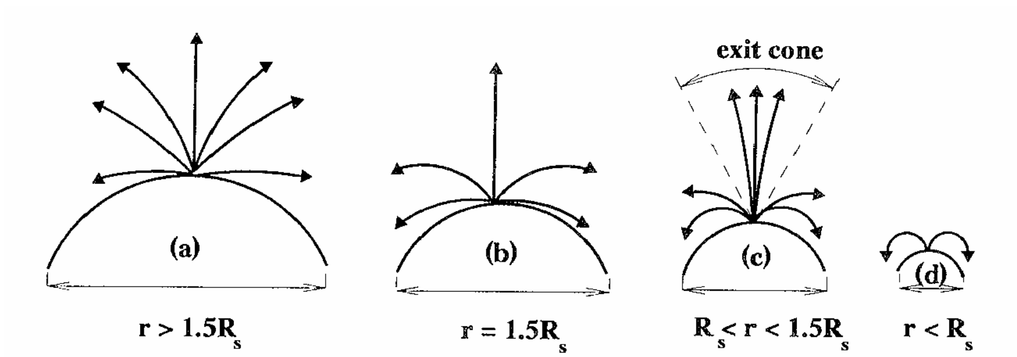
The spherical surface that marks the boundary of the black hole is called as *'event horizon'*.

* This horizon is moving out at the speed of light. Thus, in order to escape back across it, you would have to travel faster

than light. You can't go faster than light, and so you can't escape from the black hole

*** If a black hole existed, would it suck up all the matter in the Universe?**

The answer is, NO. If you cross the "horizon" of the black hole, you eventually will hit the singularity. But as long as you stay outside of the horizon, you can avoid getting sucked in.

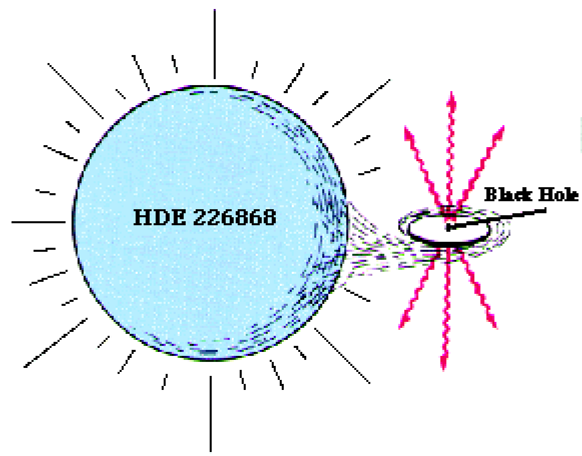


* Light will escape from any star with radius **r** greater than **1.5 R_s**. This is the radius of the **photon sphere**, and light emitted tangentially at this radius will orbit the star. If **r** is between the photon sphere and the Schwarzschild radius there will be a limited **exit cone** for escaping light, rays outside this cone fall back to the surface of the star. No light can escape from a star that has collapsed inside its own Schwarzschild radius.

❑ Is there any evidence that black holes exist?

Yes. You can't see a black hole directly, of course, since light can't get past the horizon.

* However, if a black hole passes through a cloud of interstellar matter, it can suck matter into itself. As the matter falls or is pulled towards the black hole, it gains kinetic energy and heats up. The heat ionizes the atoms, and when the atoms reach a few million degrees Kelvin, they emit X-rays. The X-rays are sent off into space before the matter crosses the Schwarzschild radius and crashes into the singularity. Thus we can see this X-ray emission.



* The Hubble Space Telescope has now provided almost conclusive evidence for the existence of black holes:

- In **1994** it found one in the heart of galaxy M87, fifty million light years from our solar system.
- In December **1995** the HST produced images of a huge disc of gas whirling around a black hole in NGC4261. It is thought to contain 1.2 billion times more matter than the Sun.