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Maximal arc partitions of designs

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Abstract

It is known that the designs $\mathrm{PG}_{n-1}(n,q)$ in some cases have spreads of maximal α-arcs. Here a α-arc is a non-empty subset of points that meets every hyperplane in 0 or α points. The situation for designs in general is not so well known. This paper establishes an equivalence between the existence of a spread of α-arcs in the complement of a Hadamard design and the existence of an affine design and a symmetric design which is also the complement of a Hadamard design.

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1. Introduction

An α-arc in a 2-design is a subset of points that meets every block in either 0 or α points. [7,8].

Rahilly [6] established the equivalence of the existence of an affine design of class number 4 and a Hadamard 2-design possessing a spread of lines of maximum size 3. By observing that a line of maximum size 3 in a Hadamard design is a 1-arc in the complementary design, we are able to extend this result and to state it in the language of maximal arcs in designs.

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2. Basic results and definitions

The general design theory used in this paper can be found in [1,3] or [4]. We shall outline in this section some definitions, notation and results.

A design $D = (\mathscr{P}, \mathscr{B}, I)$ consists of a finite point set $\mathscr{P}$, a finite block set $\mathscr{B}$, disjoint from $\mathscr{P}$ and an incidence relation $I \subseteq \mathscr{P} \times \mathscr{B}$. Where useful, we shall identify a block B with $\{p \in \mathscr{P} \mid pIB\}$; that is with the subset of points on B. Note that we do not rule out repeated blocks.

D is a t-(v, k, λ) *design* if $|\mathscr{P}| = v$, each block is on exactly k points and every subset of t points is contained in exactly λ blocks. The parameters b, r have their usual meanings: $b = |\mathscr{B}|$ and r, the replication number, is the number of blocks on any point. The number $r - \lambda$ is the *order* of a 2-(v, k, λ) design. Any t-design is also an s-design for any s, $1 \leqslant s \leqslant t$.

The dual design D^* of D is obtained by interchanging the roles of points and blocks in D. The complement $\overline{D}$ of D is defined by $\overline{D} = (\mathscr{P}, \mathscr{B}, \overline{I})$, where $\overline{I} = \mathscr{P} \times \mathscr{B} - I$. If D is a 2-(v, k, λ) design then $\overline{D}$ is a 2-$(v, v - k, b - 2r + \lambda)$ design with the same order as D.

The intersection of all blocks containing two given distinct points of D is called the *line* joining the two points. It is well-known that a line in a 2-(v, k, λ) design has at most $(b - \lambda)/(r - \lambda) = 1 + (v - 1)/k$ points, with equality if, and only if, each block either contains the line or meets it in just one point. In the case of equality, the line is said to be of *maximum size* and the line is a *maximum line.*

A set of non-empty point subsets that partitions the point set of a design is called a *spread*.

A non-empty subset S of n points of a design D is called an (α, n)*-arc* if it meets every block in at most α points. If $|B \cap S| \in \{0, \alpha\}$ for every block B of D, then S is called an α*-arc*.

Note that some authors reserve the term arc for a subset of points in a symmetric 2-design that meets any block in at most two points.

Proposition 1. *An* (α, n)*-arc A in a* 2-(v, k, λ) *design satisfies*

$$n \leqslant 1 + r(\alpha - 1)/\lambda.$$

Equality holds if and only if A is an α*-arc*.

Proof. See, e.g. [7] or [8].

A block is said to be a *secant* or *passant* of an α-arc according as it meets the arc in α or 0 points.

Let D be a 1-(v, k, r) design. Then D is *resolvable* if it has a *resolution* or *parallelism* of its block set into parallel classes, each of which partitions the point set of D. In this case, it is easy to see that D has exactly r parallel classes and each parallel class has $m = v/k$ blocks. We call m the *class number* of D. Blocks in the same parallel class are *parallel*. If the resolution is such that the intersection of any two non-parallel blocks is a constant μ, called the *index*, then D is said to be *affine*. It is straightforward to show that $\mu = k/m = k^2/v$.

A well-known theorem of Bose asserts that in a resolvable 2-(v, k, λ) design D we have $\lambda \geqslant (k - 1)/(m - 1)$, with equality if and only if D is affine. The parameters of D in the affine case can be expressed entirely in terms of μ and m as follows: $v = \mu m^2$, $k = \mu m$, $\lambda = (\mu m - 1)/(m - 1)$, $r = (\mu m^2 - 1)/(m - 1)$ and $b = rm$.

A 2-(v, k, λ) design D is *symmetric* if $b = v$. It is well-known that D is symmetric if and only if its dual design D^* is also a 2-(v, k, λ) design.

A *Hadamard* 2-*design* is a symmetric 2-(v, k, λ) design with $v=4\lambda+3$ and $k=2\lambda+1$. Such a design exists if and only if there exists a Hadamard matrix of order $v+1$. A *complementary Hadamard* 2-design is the complement of a Hadamard 2-design; so its parameters are of the form 2-$(4\lambda + 3, 2\lambda + 2, \lambda + 1)$. The Hadamard conjecture asserts that a Hadamard matrix of order n exists if and only if $n = 2$ or n is divisible by 4.

Given a Hadamard 2-$(4\lambda + 3, 2\lambda + 1, \lambda)$ design D, introduce a new point w and adjoin it to each block of D. These extended blocks and their complements give an affine 3-$(4\lambda + 4, 2\lambda + 2, \lambda)$ design. Any affine 2-design of class number 2 is in fact a 3-design obtained in this way from some (not necessarily unique) Hadamard 2-design.

The preceding discussion relating Hadamard matrices to particular classes of symmetric designs and affine designs of class number 2 is well-known. The idea has roots in a paper of Bose [2]. However, Rahilly [6] showed that there is a connection between Hadamard 2-designs and affine designs of class number 4.

Proposition 2 (*Rahilly [6]*). *There exists an affine* 2-$(16\mu, 4\mu, (4\mu - 1)/3)$ *design if and only if there exists a Hadamard* 2-$(16\mu - 1, 8\mu - 1, 4\mu - 1)$ *design with a spread of lines, all of maximum size* 3.

In this paper, we shall extend Rahilly's result to affine designs of class number m, where $m \geqslant 4$. To this end we extend the concept of lines of maximum size. One might think that this means considering, for example, plane spreads but it turns out that considering spreads of α-arcs in complements of Hadamard 2-designs leads more naturally to a generalization of Rahilly's theorem.

Rahilly's results on line spreads were for symmetric designs. We shall consider the more general theory of spreads of α-arcs in the wider setting of 2-designs, which need not be symmetric.

3. Spreads and α-arcs

First in this section, it will be shown that a line in a design D may be viewed as an α-arc in the complementary design $\overline{D}$.

Lemma 3. *Let D be a* 2-(v, k, λ) *design* $k \geqslant 3$. *Then a subset of points of D is a maximum line in $\overline{D}$ if and only if it is an α-arc in D with* $\alpha = r/(r - \lambda)$.

Proof. Let A be an α-arc in D, where $\alpha = r/(r - \lambda)$. By definition, $|A| = 1 + r(\alpha - 1)/\lambda = 1 + r/(r - \lambda)$. Therefore $|A| \geqslant 2$ and so any block of D meets A in 0 or $r/(r - \lambda)$ points; hence any block of $\overline{D}$ either contains A or meets A in exactly one point. Each of the blocks that contains two distinct points of A must therefore contain all of A and hence the line joining the two points. From the previous section, we know that a maximum line of $\overline{D}$ has exactly $1+(v-1)/(v-k)$ points, which is easily shown to equal $|A|$ using the basic design parameter relations. $\square$

Hence A is a line in $\overline{D}$. The converse is straightforward.

If A is an α-arc of D, then D_A denotes the *induced design* defined on the points of A, whose blocks are the secants of A, with induced incidence. Thus a secant B *induces* a block of D_A whose points are those of $A\cap B$. Clearly D_A is a 1-(a, α, r) design, where $|A|=a$ and r is the replication number of D. The following lemma is essentially in [8] but we include the proof for completeness.

Lemma 4. *Let A be an α-arc in a* 2-(v, k, λ) *design D. Then*

(a) *D_A is a* 2-(a, α, λ) *design, where* $a = |A| = 1 + r(\alpha - 1)/\lambda$,
(b) *A has exactly ra/α secants and $b - ra/\alpha$ passants,*
(c) *any point not in A is on exactly $\lambda a/\alpha$ secants and $r - \lambda a/\alpha$ passants,*
(d) *the passants of A form an $(r - \lambda)/\alpha$-arc in D^*.*

Proof. Condition (a) is straightforward. Moreover, for D_A the parameters ‘r’ and ‘b’ are, respectively, the replication number r of D and the number of secants of A. The standard equation ‘$bk = vr$’ then gives (b).

To prove (c) let p be a point not in A and N the number of secants on p. Counting in two ways the number of flags (q, B), where B is a secant on p and $q \in A \cap B$, gives $a\lambda = N\alpha$. Finally, (d) follows easily from (c). $\square$

Next, we consider the number of common secants and passants of two disjoint arcs.

Lemma 5. *Let A_i be an α_i-arc and $|A_i| = a_i$ for $i = 1, 2$, where $A_1 \cap A_2 = \emptyset$. Then the number of secants common to A_1 and A_2 is $\lambda a_1 a_2/\alpha_1\alpha_2$ and the number of common passants is $b - (a_1\alpha_2 + a_2\alpha_1 - \lambda a_1 a_2)/\alpha_1\alpha_2$.*

Proof. Let x be the number of common secants. Counting in two ways the number of ordered triples (p_1, p_2, B), where $p_i \in A_i$ and B is a block containing p_i $(i = 1, 2)$, gives $a_1a_2\lambda = x\alpha_1\alpha_2$. The rest is straightforward using this result and Lemma 4. $\square$

Remark 6. Rahilly [6] defines a spread of maximum lines to be *uniform* if the number of blocks containing any two lines of the spread is constant. He then proves that every spread of maximum lines in a Hadamard 2-design is uniform. However, this is true for all 2-designs as can easily be deduced from Lemmas 3 and 5.

The *mth multiple* design of a design is obtained by repeating each of its blocks m times.

The case when the induced design on an α-*arc* is a multiple of a symmetric design is of special interest. Let D be a 2-(v, k, λ) design with an α-*arc* A. Then D_A is a 2-(a, α, λ) design, where $a = 1 + r(\alpha - 1)/\lambda$ and the replication number of D_A is r, that of D. Hence if D_A is a multiple of a symmetric design, then it is the (r/α)th multiple of a symmetric 2-(a, α, λ') design denoted by $[D_A]$, where $\lambda' = \lambda\alpha/r$. In this case we shall say that A is a *symmetric α-arc*.

A set of α-*arcs* that partitions the point set of D will be called an α-*spread*. If all the α-*arcs* in the spread are symmetric, it is called a *symmetric* α-*spread*.

In view of Lemma 3, every $r/(r-\lambda)$-spread in D is a line spread in $\overline{D}$ in the sense of Rahilly [6]: that is a partition of the point set by maximum lines. We shall show that in the case $\alpha = r/(r-\lambda)$, all α-arcs and α-spreads are symmetric.

Lemma 7. *Every* $[r/(r-\lambda)]$*-arc in a* 2-(v, k, λ) *design is symmetric and is a maximum line in the complementary design.*

Proof. First note that if x is a point of a maximum line of a 2-(v, k, λ) design, the number of blocks containing x but not the whole line is $r-\lambda$, the order of the design.

Now suppose A is an α-arc of a 2-(v, k, λ) design D, where $\alpha = r/(r-\lambda)$. Then $|A| = 1+\alpha$ and D_A is a 2-$(\alpha+1, \alpha, \alpha-1)$ design. By Lemma 3, A is a maximum line in $\overline{D}$. Therefore, given a point of A, the number of blocks of $\overline{D}$ meeting A only at that point is the order of $\overline{D}$, which is the same as the order $r-\lambda = r/\alpha$ of D. Hence each block of D_A is repeated r/α times and so A is a symmetric α-arc. □

Theorem. *There exists an affine* 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ *design and a complementary Hadamard* 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ *design if and only if there exists a complementary Hadamard* 2-$(\mu m^2-1, \frac{1}{2}\mu m^2, \frac{1}{4}\mu m^2)$ *design with a symmetric* $\frac{1}{2}m$*-spread.*

Proof. First assume there exists an affine 2-$(\mu m^2, \mu m, (\mu m-1)/(m-1))$ design Γ and a 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ design Δ.

Choose a point w of Γ. Then on the remaining μm^2-1 points of Γ define a design Π whose blocks are obtained thus. For each parallel class C of Γ, identify the $m-1$ blocks of C not on w with the points of Δ. Then the union of the $\frac{1}{2}m$ blocks of Γ corresponding to a block of Δ is defined to be a block of Π.

Hence Π has μm^2-1 points and $\mu m \times \frac{1}{2}m = \frac{1}{2}\mu m^2$ points on each block. To evaluate the replication number of Π, let x be any of its points. There are '$r-\lambda$' $= \mu m$ parallel classes of C of Γ such that x and w are on different blocks from C.

The block of C on x, considered as a point of Δ, is in $\frac{1}{2}m$ blocks of Π. Hence x is on $\frac{1}{2}m$ blocks of Π induced by C. Therefore, in total, x is on $(\frac{1}{2}m) \times (\mu m) = \frac{1}{2}\mu m^2$ blocks of Π. It follows that Π is a symmetric design since '$r = k$'.

Now consider two distinct blocks X and Y of Π. If they are induced by the same parallel class C of Γ, then from the parameters of Δ it follows that X and Y have $\frac{1}{2}m$ blocks of C in common and therefore meet in $(\frac{1}{4}m) \times (\mu m) = \frac{1}{4}\mu m^2$ points of Π.

Suppose on the other hand, that X and Y are induced by different parallel classes of Γ. Since X and Y each consists of $\frac{1}{2}m$ blocks of Γ and non-parallel blocks of Γ meet in μ points, it follows that X and Y meet in exactly $\mu \times (\frac{1}{2}m)^2 = \frac{1}{4}\mu m^2$ points of Π.

Hence the dual of Π is a symmetric 2-design. Therefore Π and its dual Π^* are symmetric 2-designs with parameters 2-$(\mu m^2-1, \frac{1}{2}\mu m, \frac{1}{4}\mu m)$.

Next, we show that Π^* has a symmetric $\frac{1}{2}m$ spread. Let C be any parallel class of Γ and x any point of Π. Let X be the block of C on x. If also w is on X, then no block of Π induced by C contains x. Otherwise the number of blocks on x induced by C is the number of blocks containing X (considered as a point of Δ) which is therefore the replication number $\frac{1}{2}m$ of Δ. Hence the $m-1$ blocks of Π induced by C form an α-arc in Π^*, where $\alpha = \frac{1}{2}m$. We show this arc is symmetric, noting here that $r/\alpha = \frac{1}{2}\mu m^2/\frac{1}{2}m = \mu m$.

In the case when x is on $\frac{1}{2}m$ blocks of Π (induced by C), all the μm points of X are on the same $\frac{1}{2}m$ blocks. This shows that the $m-1$ blocks induced by C form a symmetric $\frac{1}{2}m$-arc in Π^*.

Clearly, by varying C over all parallel classes of Γ, we obtain a symmetric $\frac{1}{2}m$-spread in Π^*.

Conversely, assume the existence of a 2-$(\mu m^2-1, \frac{1}{2}\mu m^2, \frac{1}{4}\mu m^2)$ design D with a symmetric $\frac{1}{2}m$-spread Σ. Let $A \in \Sigma$. Then A is a symmetric $\frac{1}{2}m$-arc. Further, by Lemma 4, $|A| = m-1$, A has $\mu m(m-1)$ secants and $\mu m - 1$ passants. Since A is a symmetric $\frac{1}{2}m$-*arc* it follows easily that D_A is a symmetric 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ design.

Define a design Γ as follows. The points of Γ are those of D^* and a new point, labelled w. The blocks of Γ are of two types. Type 1 blocks are labelled $\langle A\rangle$, $A \in \Sigma$. Hence there are $(\mu m^2-1)/(m-1)$ blocks of Type 1.

Type 2 blocks of Γ are labelled $\langle A, e\rangle$, where $A \in \Sigma$ and e is any block of $[D_A]$. Hence since $|\Sigma| = (\mu m^2-1)/(m-1)$ and each $[D_A]$ has $m-1$ blocks, it follows that there are $\mu m^2 - 1$ blocks of Type 2. Therefore Γ has exactly $m(\mu m^2-1)/(m-1)$ blocks.

Finally to complete the definition of Γ, we define incidence in Γ.

(i) If $A \in \Sigma$, then $\langle A\rangle$ is incident with w and with all the passants of A in D: they are points of D^* and therefore of Γ. By Lemma 4, $\langle A\rangle$ is on exactly $1+(\mu m-1) = \mu m$ points.

(ii) Let $\langle A, e\rangle$ be a Type 2 block as defined above. Each block e of $[D_A]$ is the intersection with A of any one of μm secants of A in D, since A is symmetric; so that each block of D_A is repeated 'r/α' times. (Here $r = \frac{1}{2}\mu m^2$ and $\alpha = \frac{1}{2}m$.) These μm secants as points of D^* are defined to be incident with $\langle A, e\rangle$ in Γ.

Hence Γ has μm^2 points, with μm points on each block. Next, we show Γ is a 2-design. Consider two distinct points X and Y of Γ. There are two cases.

Case 1: $Y = w$. Then only Type 1 blocks contain X and Y and the number of such blocks is the number ρ of $A \in \Sigma$ for which Y is a passant in D. Since Σ partitions the points of D and Y is a secant to $(\mu m^2-1)/(m-1) - \rho$ of the $\frac{1}{2}m$-arcs in Σ, then $(\mu m^2-1)/(m-1) - \rho = (\frac{1}{2}\mu m^2)/(\frac{1}{2}m) = \mu m$, whence $\rho = (\mu m-1)/(m-1)$.

Case 2: Neither X nor Y is w. Let π be the number of $A \in \Sigma$ such that X and Y are both passants of A in D. Then exactly $\sigma = (\mu m^2-1)/(m-1) - 2\rho + \pi$ of the arcs $A \in \Sigma$ are such that X and Y are both secants of A. Furthermore, π is the number of Type 1 blocks of Γ containing both X and Y.

Let τ be the number of Type 2 blocks of Γ containing X and Y. We need to evaluate $\pi + \tau$. First observe that X and Y are both secants to exactly σ of the arcs in Σ. That is they induce the same block in τ of the symmetric 2-$(m-1, \frac{1}{2}m, \frac{1}{4}m)$ designs $[D_A]$, and induce different blocks in the $\sigma - \tau$ remaining $[D_A]$, where $A \in \Sigma$ and X, Y are both secants of A. That is, for τ of the arcs $A \in \Sigma$, the blocks $A \cap X$ and $A \cap Y$ of $[D_A]$ are equal, so that $|A \cap X \cap Y| = |A \cap X| = \frac{1}{2}m$; while for $\sigma - \tau$ of the arcs, $A \cap X$ and $A \cap Y$ meet in $\frac{1}{4}m$ points, so that $|A \cap X \cap Y| = \frac{1}{4}m$. For the remaining $A \in \Sigma$, either X or Y is a passant, so that $A \cap X \cap Y = \phi$.

Since from the parameters of the symmetric design D we have $|X \cap Y| = \frac{1}{4}\mu m^2$, it follows that $\frac{1}{4}\mu m^2 = \frac{1}{2}m\tau + \frac{1}{4}m(\sigma - \tau)$, whence $\mu m = \sigma + \tau$. Substituting for σ and ρ we obtain $\pi + \tau = (\mu m - 1)/(m - 1) = \rho$.

It follows that Γ is a 2-$(\mu m^2, \mu m, (\mu m - 1)/(m - 1))$ design. A straightforward check will verify that Γ is resolvable: a typical parallel class is given by each $A \in \Sigma$ and consists of the block $\langle A \rangle$ together with the $m - 1$ blocks $\langle A, e \rangle$, where e is any of the $m - 1$ blocks of $[D_A]$. Hence from Bose's theorem (see Section 1) it follows that Γ is affine. $\square$

As a corollary we can readily obtain the proposition due to Rahilly [6] stated earlier. Since a 2-(3, 2, 1) design always exists, then for $m = 4$ the above theorem states that the existence of an affine 2-$(16\mu, 4\mu, \frac{1}{3}(4\mu - 1))$ design is equivalent to the existence of a complementary Hadamard 2-$(16\mu - 1, 8\mu, 4\mu)$ design with a symmetric 2-spread. Now apply Lemma 3.

An interesting case is $m=4$, $\mu=7$. Then the theorem implies that the existence of an affine 2-(112, 28, 9) design is equivalent to the existence of a Hadamard 2-(111, 55, 27) design with a spread of lines, all of size 3. The existence of such an affine design is undecided. According to Tonchev, it is the smallest undecided affine 2-design: on the other hand, there exist Hadamard designs on 111 points but it is not known whether any of them have spreads.

Examples of spreads of α-arcs are to be found in the designs $\mathrm{PG}_{n-1}(n, q)$ of the points and hyperplanes in $\mathrm{PG}(n, q)$. If $t + 1$ divides $n + 1$, then $\mathrm{PG}_{n-1}(n, q)$ contains a spread of t-dimensional subspaces which in the complementary design is a symmetric q^t-spread. See, e.g. [3].

Jungnickel and Tonchev [5] showed that there exist symmetric designs with the parameters of, but not isomorphic to $\mathrm{PG}_{n-1}(n, q)$, namely GMW designs, possessing spreads of α-arcs.

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