Proc 2nd Saudi Sci. Conf.,Fac. Sci., KAU, 15-17 March 2004, ***Part II***, pp. 31-43 (2005 A.D./1425 A.H.)

Mass Spectrum of Heavy-Quark Hadrons

M. ARAFAH and AFAF BAABDULLAH
Physics Department, Faculty of Sciences, King Abdulaziz University, Jeddah, Saudi Arabia

ABSTRACT. The spectrum of the upsilon meson system Υ consists of bound states of a bottom quark b and anti-bottom quark $\bar{b}$ called bottomonium. Upsilon is a massive meson of about 9.46GeV. It was discovered by resonance peaks produced at certain energies in experiments involving the production of lepton pairs. And the higher energy peaks are the excited states of Υ.

In this work, we calculate the mass spectrum of this system, by using Dirac equation, which is a relativistic equation, and holds for spin 1/2 particles. We assume two suitable models of interaction potentials between the constituent quarks of Υ system The problem is solved by using WKB-method.

We develop a computer program to obtain the energy eigenvalues for different values of K quantum number and for n = 0, 1, 2 and 3. Finally, we compare our results with experimental values, to show a good agreement especially for $K = 1$.

1. Introduction

The recent advent (1995) of the heavy quark called t-quark (top quark) is a great success to the quark theory of elementary particles inviting new research projects that could be developed for deep investigation of the origin of matter.

The quark model was introduced by Gell-Mann and Zweig (1964). The biggest success of the quark model was the discovery of the Omega minus (Ω^-) particle which was predicted previously by this model. Since then this model has been developed by many scientist. In literature one finds several attempts of describing hadronic properties in terms of the interaction between quarks using the Schrödinger equation (Arafah,1981; Gue, 2003; Manohar *et al.*, 2000). On the other hand the quarkonium system may be solved relativistically as a one-body problem by solving Dirac equation with a linear scalar potential using the method of infinite series, in which case there appears a three-term recursion relation.

Ram and Halasa (1979) applied the same method of infinite series to a quadratic scalar potential which leads to a four-term recursion relation. The solution of the four – term recursion relation gave complex energy eigenvalues for the quadratic scalar potential. Ram (1980) pointed out that when the Dirac equation is transformed to an equivalent

Schrödinger equation, the latter gives real energy eigenvalues for the quadratic scalar potential just as it does for the linear potential.

In recent years, much work has been done in this field. For example, Casanova (2001) studied the quark-quark correlation using Dirac equation. Gue Jian-You (2003) solved the Dirac equation with special potential. Also Avila (1999) worked on solutions of a Dirac H-like meson and scalar confinement potential for low angular momentum state.

In the present work we study the energy eigenvalues resulting from Dirac equation with scalar potential using the WKB method. In particular, we calculate the energy eigenvalues for the upsilon system Υ, which is a bound state of a b-quark and anti b-quark.

2. Solving the Dirac Eqnation with a Scalar Potential

With the customary definition of Dirac matrices $\vec{\alpha}$ and β , the Dirac equation is given by:

$$H\psi_{jm\omega} = [\vec{\alpha}\cdot\vec{p} + \beta m(r)]\psi_{jm\omega} \qquad \ldots (1)$$

where $\hbar = c = 1$ and the spinor is

$$\psi_{jm\omega} = \begin{pmatrix} \psi_{l_1 jm} \\ \psi_{l_2 jm} \end{pmatrix} = \frac{1}{r}\begin{pmatrix} F(r)\; Y_{l_1 jm} \\ iG(r)\; Y_{l_2 jm} \end{pmatrix} \qquad \ldots (2)$$

with

$$m(r) = \mu + V_s(r). \qquad \ldots (3)$$

V_s represents a scalar potential and μ is the reduced mass.

In equation (2), j denotes the total angular momentum operator

$$l_1 = j + \frac{1}{2}\omega,\;\; l_2 = j - \frac{1}{2}\omega \;\; and \;\; \omega = \pm 1 \;. \qquad \ldots (4)$$

Using the standard procedure, it is straight forward to show that the eigenvalue equation $H\psi_{jm\omega} = E\psi_{jm\omega}$ separates into the two following coupled equations

$$\left.\begin{aligned} &(E - m(r))F(r) + G'(r) - ((j+1/2)\omega/r)G(r) = 0\;. \\ &(E + m(r))G(r) - F'(r) - ((j+1/2)\omega/r)F(r) = 0\;. \end{aligned}\right\} \qquad \ldots (5)$$

In order to be able to use equation (5) for both cases $(l = j \pm \frac{1}{2})$, it is convenient to write

$$\kappa = \begin{cases} (j+\frac{1}{2}) = l+1 & for\; j = l+\frac{1}{2}\;, \\ -(j+\frac{1}{2}) = -l & for\; j = l-\frac{1}{2}\;. \end{cases} \qquad \ldots (6)$$

The number κ takes all integral values except zero, the positive numbers corresponding to the case $(l = j+\frac{1}{2})$ and negative numbers to the case $(l = j-\frac{1}{2})$. Introducing

$$F(r) = r\psi_a \text{ and } G(r) = r\psi_b, \qquad \ldots (7)$$

and using eqs.(6) and (7) we obtain

$$(E - m(r))\psi_a + \frac{d\psi_b}{dr} + \frac{\kappa+1}{r}\psi_b = 0\;, \qquad \ldots (8a)$$

$$(E + m(r))\psi_b - \frac{d\psi_a}{dr} + \frac{\kappa - 1}{r}\psi_a = 0 \quad . \qquad \ldots (8b)$$

where ψ_a and ψ_b are called the large and small components of the spinor (2) and $m(r) = \mu + V_s(r)$.

The transformation

$$\psi_1 = \frac{r\psi_a}{\sqrt{V_s + \mu + E}} \quad , \qquad \ldots \ (9a)$$

$$\psi_2 = \frac{r\psi_b}{\sqrt{V_s + \mu - E}} \quad . \qquad \ldots \ (9b)$$

transforms equations (24a) and (24b) into the following set of equations,

$$\frac{d^2\psi_1}{dr^2} + (E^2 - \mu^2 - V_{eff}^1)\psi_1 = 0 \ , \qquad \ldots \ (10a)$$

$$\frac{d^2\psi_2}{dr^2} + (E^2 - \mu^2 - V_{eff}^2)\psi_2 = 0 \qquad \ldots \ (10b)$$

with

$$V_{eff}^1 = \frac{\kappa(\kappa - 1)}{r^2} + V_s^2 + 2\mu V_s - \frac{1}{2}\frac{V_s'' + 2V_s' \ (\kappa / r)}{(\mu + V_s + E)} + \frac{3}{4}\frac{(V_s' \ )^2}{(\mu + V_s + E)^2} \qquad \ldots \ (11)$$

and

$$V_{eff}^2 \ (\kappa, E) = V_{eff}^1 \ (-\kappa, -E). \qquad \ldots \ (12)$$

In equation (11) primes mean differentiations with respect to r.
The energy eigenvalues E in equation (10a) have been obtained by using the WKB method,

$$\int_a^b k(x)dx = (n + \frac{1}{2})\pi, \qquad n = 0, 1, 2, \ldots, \qquad \ldots \ (13)$$

where k(x) is the wave number.

3. Potential Models and Calculation

In this section two scalar potential models were studied using the Dirac equation, to fit the energy levels of the Upsilon system (bottomonium) applying the technique developed earlier.

The Upsilon system Υ is a bound state of a bottom quark and anti-bottom quark $\bar{b}$, quarks with charge 1/3 and mass 4.5 GeV. Various excited states (resonances) of this system can be created in $e^+ e^-$ experiments at certain energies.

Model I

In the first model we assume a linear potential plus the Coulomb potential:

$$V_s = \lambda_1 r - \alpha / r + c_1 \qquad \ldots \quad (14)$$

where λ_1 , α *and* c_1 are the potential parameters.
With this potential V_{eff}^1 equation (11) reduces to

$$V_{eff}^1 = \frac{\kappa(\kappa-1)}{r^2} + (\lambda_1 r - \frac{\alpha}{r} + c_1)^2 + 2\mu(\lambda_1 r - \frac{\alpha}{r} + c_1) - \frac{1}{2}\frac{\left(\frac{-2\alpha}{r^3} + 2\kappa\frac{(\lambda_1 + \alpha/r^2)}{r}\right)}{(E + \mu + \lambda_1 r - \alpha/r + c_1)} + \frac{3}{4}\frac{(\lambda_1 + \alpha/r^2)^2}{(E + \mu + \lambda_1 r - \alpha/r + c_1)^2} \qquad (15)$$

V_{eff}^1 for this potential for certain energy is shown in Figure (1), the curve shows that we have a bound state system.
The equivalent Schrödinger equation with this potential is

$$\frac{d^2\psi_1}{dr^2} + k_{ll}^2(r)\,\psi_1 = 0 \ . \qquad \ldots \quad (16)$$

and it is solved by using the WKB method

$$\int_a^b k_{ll}(r)\,dr = \left(n + \frac{1}{2}\right)\pi, \qquad n = 0,1,2,\ldots, \qquad \ldots \quad (17)$$

where a and b are the roots of $k_{ll}(r) = 0$, and

$$k_{ll}(r) = \left(E^2 - \mu^2 - V_{eff}^1\right)^{1/2}$$

$$= \left(E^2 - \mu^2 - \left(\frac{\kappa(\kappa-1)}{r^2} + (\lambda_1 r - \frac{\alpha}{r} + c_1)^2 + 2\mu(\lambda_1 r - \frac{\alpha}{r} + c_1) - \frac{1}{2}\frac{\left(\frac{-2\alpha}{r^3} + 2\kappa\frac{(\lambda_1 + \alpha/r^2)}{r}\right)}{(E + \mu + \lambda_1 r - \alpha/r + c_1)} + \frac{3}{4}\frac{(\lambda_1 + \alpha/r^2)^2}{(E + \mu + \lambda_1 r - \alpha/r + c_1)^2}\right)\right)^{1/2}$$

(18)

The quantity $k_{ll}^2(r)$ for this potential model at certain energy is shown in Figure(2) .

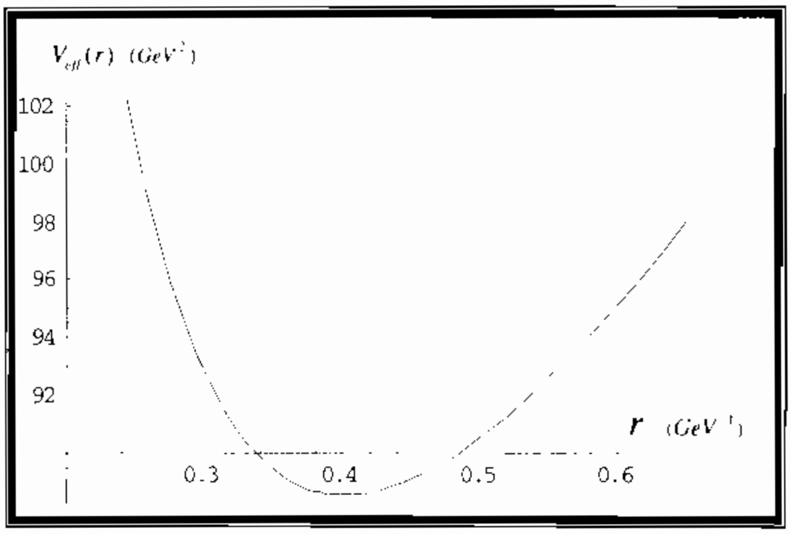

Fig. (1). V_{eff}^1 with scalar potential $V_s = \lambda_1 r - \alpha / r + c_1$, represents a bound state system, for κ = -1, n = 0, and E = 9.8599 GeV

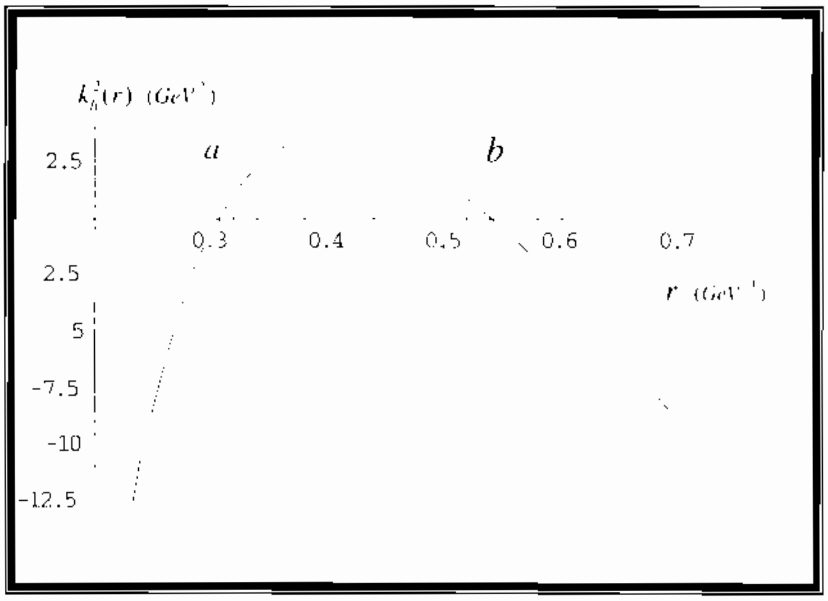

Fig. (2). $k_b^2(r)$ with scalar potential $V_s = \lambda_1 r - \alpha / r + c_1$, shows the roots a and b, for κ = -1, n = 0, and E = 9.8599 GeV.

We calculate the energy eigenvalues for $b\,\bar{b}$, the results are shown in Table (1)

Table (1). Mass spectrum for $b\bar{b}$ systems resulting from the Dirac equation for the levels $|\kappa| \le 3$ and n = 0, 1, 2, and n=0, 1, 2,and 3. E represents our solutions with $V_s = \lambda_1 r - \alpha / r + c_1$ when m_b = 4.5 GeV. $\lambda_1 = 3.9\, GeV^2$, $\alpha = 0.01$ *and* $c_1 = 5.1\, GeV$, where the values of a and b are the roots of $k_n^2(r)$. And E (Arafah) represents the solutions when m_b = 4.94 GeV. $\lambda_1 = 0.0771\, GeV^2$, $\alpha = 0.473$.

$\lambda_1 = 3.9\, GeV^2$, $\alpha = 0.01$, $c_1 = 5.1\, GeV$						
κ	n	a	b	E (Present work) GeV.	E (Arafah) Previous work GeV.	(Experimental energy) GeV.
1	0	0.0080	0.513	9.3099	9.2527	9.46030
	1	0.0075	0.70	10.0418	10.033	10.02326
	2	0.0072	0.803	10.4600	10.4023	10.3552
	3	0.0070	0.881	10.7675	10.6745	10.5800
-1	0	0.279	0.602	10.0142	9.9133	9.8599
	1	0.235	0.748	10.4621	10.3104	9.8927
	2	0.215	0.841	10.7797	10.5946	9.9126
	3	0.202	0.913	11.0322	10.8259	
2	0	0.272	0.597	9.9502	9.8856	10.2321
	1	0.228	0.745	10.4142	10.2926	10.2552
	2	0.209	0.839	10.7395	10.5802	10.2685
	3	0.197	0.910	10.9968	10.8170	
-2	0	0.443	0.703	10.7124	10.2158	
	1	0.388	0.819	10.9876	10.5107	
	2	0.359	0.899	11.2133	10.7526	
	3	0.340	0.961	11.4067	10.9638	
3	0	0.435	0.696	10.6360	10.2052	
	1	0.380	0.814	10.9226	10.5022	
	2	0.352	0.894	11.01557	10.7451	
	3	0.333	0.957	11.3543	10.9570	
-3	0	0.583	0.804	11.3725	10.4232	
	1	0.525	0.901	11.5524	10.6727	
	2	0.493	0.968	11.7122	10.8894	
	3	0.470	1.023	11.8566	11.0844	

Model II

In the second model we assume a quadratic potential plus the Coulomb potential:

$$V_s = \lambda_2 r^2 - \alpha / r + c_2 \qquad \ldots \ (19)$$

Where λ_2 , α *and* c_2 are the potential parameters.

V_{eff}^1 equation (11) reduces to

$$V_{eff}^1 = \frac{\kappa(\kappa-1)}{r^2} + (\lambda_2 r^2 - \frac{\alpha}{r} + c_2)^2 + 2\mu(\lambda_2 r^2 - \frac{\alpha}{r} + c_2) - \frac{1}{2}\frac{\left(2\lambda_2 - \frac{2\alpha}{r^3} + \frac{2\kappa(2r\lambda_2 + \alpha / r^2)}{r}\right)}{(E + \lambda_2 r^2 - \frac{\alpha}{r} + c_2 + \mu)}$$

$$+ \frac{3}{4}\frac{(\frac{\alpha}{r^2} + 2\lambda_2 r)^2}{(E + \lambda_2 r^2 - \frac{\alpha}{r} + c_2 + \mu)^2}$$

(20)

V^1_{eff} for this potential for certain energy is shown in Figure (3), the curve shows that we have a bound state system.

The equivalent Schrödinger equation with this potential is

$$\frac{d^2\psi_1}{dr^2}+k^2_{qu}(r)\,\psi_1=0 \qquad \dots\ (21)$$

Again, it is solved by using the WKB method

$$\int_a^b k_{qu}(r)\,dr=\left(n+\frac{1}{2}\right)\pi, \qquad n=0,1,2,..., \qquad \dots\ (22)$$

where a and b are the roots of $k_{qu}(r)=0$, and

$$k_{qu}(r)=\left(E^2-\mu^2-V^1_{eff}\right)^{1/2}$$

$$=\left(E^2-\mu^2-\left(\frac{\kappa(\kappa-1)}{r^2}+(\lambda_2 r^2-\frac{\alpha}{r}+c_2)^2+2\mu(\lambda_2 r^2-\frac{\alpha}{r}+c_2)-\frac{1}{2}\frac{\left(2\lambda_2-\frac{2\alpha}{r^3}+\frac{2\kappa(2r\lambda_2+\alpha/r^2)}{r}\right)}{(E+\lambda_2 r^2-\frac{\alpha}{r}+c_2+\mu)}+\frac{3}{4}\frac{(\frac{\alpha}{r^2}+2\lambda_2 r)^2}{(E+\lambda_2 r^2-\frac{\alpha}{r}+c_2+\mu)^2}\right)\right)^{1/2}$$

(23)

Also, $k^2_{qu}(r)$ for this potential model at certain energy is shown in Figure (4).

We also fix the parameters λ_2, α *and* c_2; and apply the same technique as above to calculate the energies for this set of the parameters. The results are shown in Table (2).

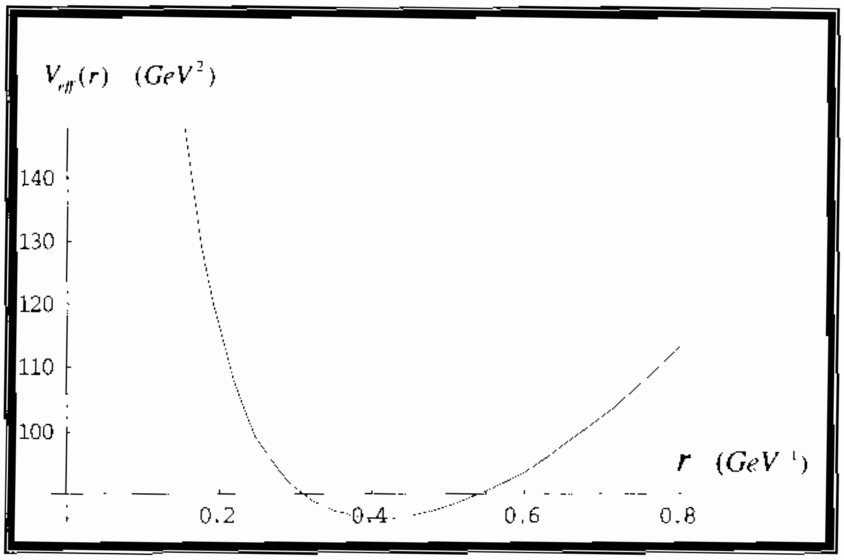

Fig. (3). V^1_{eff} with scalar potential $V_s = \lambda_2 r^2 - \alpha / r + c_2$, represents a bound state system, for κ = -1, n = 0, and E = 9.8599 GeV.

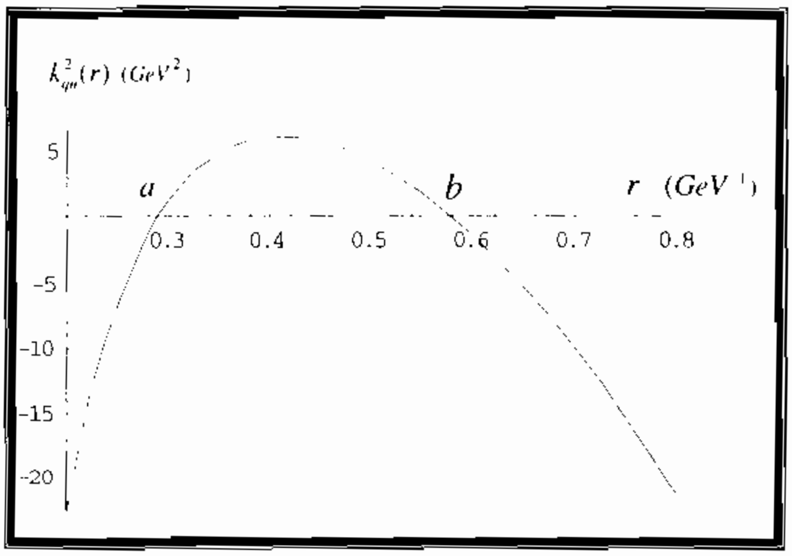

Fig. (4). $k^2_{qm}(r)$ with scalar potential $V_s = \lambda_2 r^2 - \alpha / r + c_2$, shows the roots a and b, for κ = -1, n = 0, and E = 9.8599 GeV.

Table (2). Mass spectrum for $b\bar{b}$ systems resulting from the Dirac equation for the levels $|\kappa| \le 3$ and n = 0, 1, 2, and 3. E represents our solutions with $V_s = \lambda_2 r^2 - \alpha / r + c_2$ when m_b = 4.5 GeV, λ_2=3.9 GeV^3, α=0.01 *and* c_2=6GeV, where the values of a and b are the roots of $k^2_{sph}(r)$. And E (Arafah) represents the solutions when m_b = 4.94 GeV, λ_2=0.0771 GeV^3, α=0.473.

$\lambda_2 = 3.9\ GeV^3,\ \alpha = 0.01,\ c_2 = 6\ GeV$						
κ	n	a	b	**E (present work) GeV.**	**E (Arafah) Previous work GeV.**	**Experimental energy GeV.**
1	0	0.0094	0.524	9.2693	9.2527	9.46030
	1	0.0084	0.685	10.0398	10.033	10.02326
	2	0.0079	0.775	10.5560	10.4023	10.3552
	3	0.0075	0.840	10.9683	10.6745	10.5800
-1	0	0.295	0.599	9.9653	9.9133	9.8599
	1	0.241	0.724	10.4985	10.3104	9.8927
	2	0.215	0.802	10.9195	10.5946	9.9126
	3	0.199	0.861	11.2767	10.8259	
2	0	0.294	0.599	9.9030	9.8856	10.2321
	1	0.240	0.724	10.4425	10.2926	10.2552
	2	0.215	0.802	10.8677	10.5802	10.2685
	3	0.198	0.861	11.2281	10.8170	
-2	0	0.430	0.673	10.6989	10.2158	
	1	0.371	0.774	11.0818	10.5107	
	2	0.339	0.841	11.4144	10.7526	
	3	0.316	0.893	11.7118	10.9638	
3	0	0.430	0.673	10.6076	10.2052	
	1	0.370	0.773	10.9969	10.5022	
	2	0.338	0.840	11.3345	10.7451	
	3	0.316	0.892	11.6359	10.9570	
-3	0	0.536	0.743	11.4486	10.4232	
	1	0.478	0.827	11.7377	10.6727	
	2	0.444	0.885	12.0023	10.8894	
	3	0.420	0.931	12.2473	11.0844	

4. Conclusion

From the calculated energies of model I, which are listed in Table (1) we find the following. We have a good agreement between the theoretical results and the experimental results for $\kappa = 1$ in the states n = 0, 1, 2 and 3, with average percentage error 1.14 %, for $\kappa = -1$ in the state n = 0, with percentage error 1.5%, and for $\kappa = 2$ in the states n = 0, 1 and 2, with average percentage error 2.15 %., But we do not have a good agreement between the theoretical results and experimental results for $\kappa = -1$ in the states n = 1, and 2 with average percentage error 7.25%. For $\kappa = -2, 3, -3$, we compare our present results with previous theoretical results, and the average percentage error is about 5.3%.

Figure (5) shows the mass spectrum of the $b\bar{b}$ system of the potential model I compared to the experimental values.

From the calculated energies of model II, which are listed in Table (2) we find the following. We have a good agreement between the theoretical results and the experimental results for $\kappa = 1$ in the states n = 0, 1, 2 and 3, with average percentage error 1.95 %, for $\kappa = -1$ in the state n = 0, with percentage error 1.07 %, and for $\kappa = 2$ in the states n = 0, and 1 with average percentage error
2.5 %. But we do not have a good agreement between the theoretical results and experimental results for $\kappa = -1$ in the states n = 1, and 2 with average percentage error 8.1%, and for $\kappa = 2$ in the state n = 2 with average percentage error 5.83 %.

For $\kappa = -2, 3, and\ -3$, we compare our present results with previous theoretical results, and the average percentage error is about 6.9 %.
Figure (6) shows the mass spectrum of the $b\bar{b}$ system of the potential model II compared to the experimental values.
From above discussion we conclude that the two assumed models of the confining potentials of $b\bar{b}$ meson fit the eigenvalues and the corresponding mass spectra very well with the experimental data, and the predicted energy levels for $\kappa = -2, 3, and\ -3$ give a good agreement with other values obtained from different quark binding potentials.
Dirac equation is useful to study heavy-light mesons and to describe them as an atom-like system. Beside this advantage, Dirac equation may describe any composite system with any number of constituents, such as baryons.

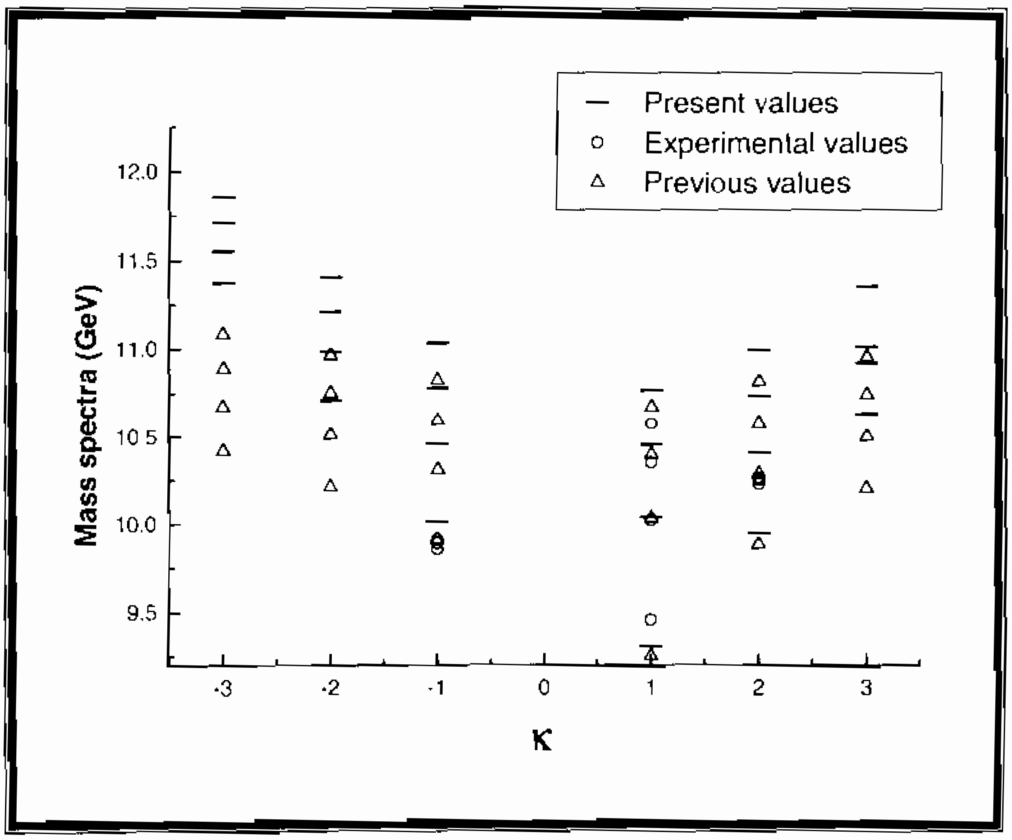

Fig. (5). Mass spectra for $b\bar{b}$ system resulting from the Dirac equation in potential model I $V_s = \lambda_1 r - \alpha / r + c_1$ (horizontal lines) compared to the experimental values and previous results based on Table (5-1).

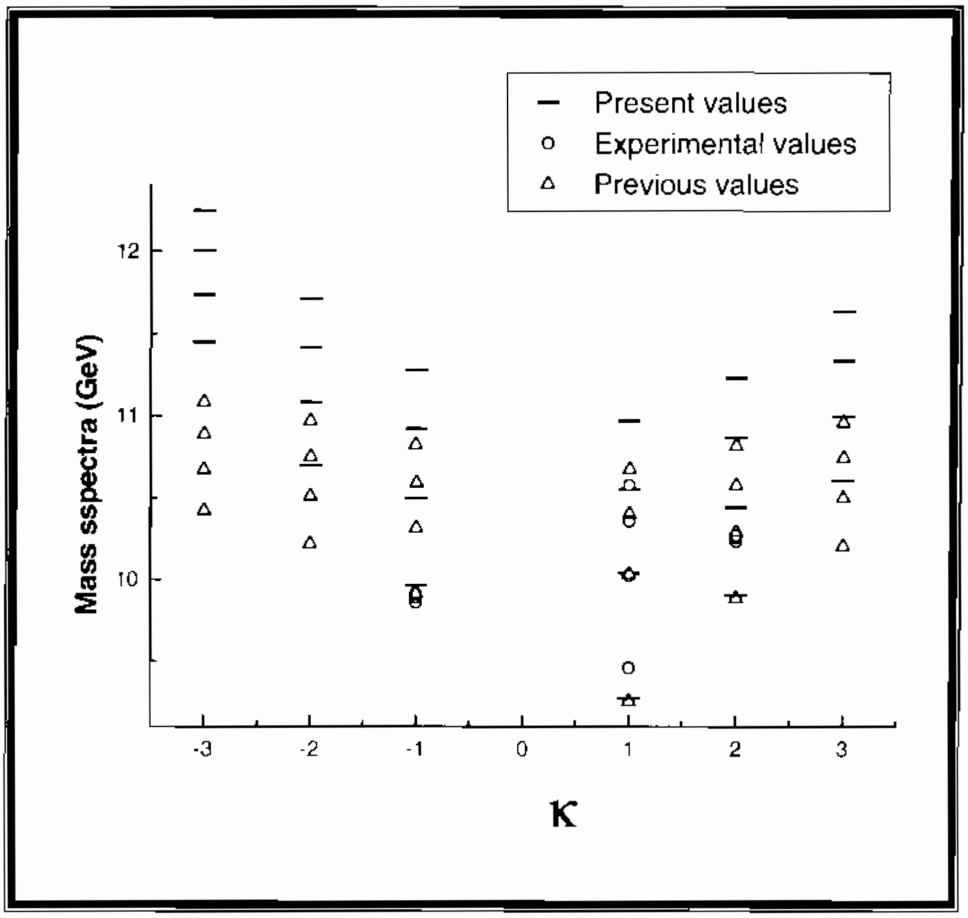

Fig. (6). Mass spectra for $b\bar{b}$ system resulting from the Dirac equation in potential model II $V_s = \lambda_2 r^2 - \alpha / r + c_2$ (horizontal lines) compared to the experimental values and previous results based on Table (2).

References

Arafah, M. ***et al.*** (1980), Lett. Al Nuovo Cimento 30, 5-8.

Arya, A.P. (1974) *Elementary modern physics,* Canada: Addison-Wesley publishing company, p. 401-402, 484, 486-489, 491-493, 500-504, and 508-512.

Anderson, E.E. (1971) *Modern physics and quantum mechanics,* Philadelphia: W. B. Saunders Company, p. 371-376.

Avila, M. (1999), *Lett. international journal of modern physics* A **14,** 1703-1710.

Casanova, S. (2001) *Analysis of non-forward quark-quark correlator.* Thesis, submitted for the degree of Doctor of Philosophy in physics at the universities of Pavia and Wuppertal.

Cheng, D.C. , O'Neil, and **Gerard K.** (1979) *Elementary particle physics: An Introduction,* Canada: Addison-Wesley published company, p. 5-7.

Gue Jian-you (2003) solution of the, *Lett. Chinese physics* **20,** 602-604.

Hagiwara, K. ***et al.*** **Particle data group,** (July 2002) *Particle physics booklet,* CERN, p.100-104, and 141.

Khanna, M.P. (1999) *Introduction to particle physics,* New Delhi: Prentice-Hall of India, p. 2-11, 15-19, 76-78, 117, 125, 222, and 244.

Kobayashi, M. and **Maskawa, T.** (1973), *Lett. Progress of theoretical physics* **49,** 652.

Manohar, A.V., Wise, and **Mark B.** (2000) *Heavy quark physics,* Cambridge: Cambridge university press, p. 45-50.

Ram, B. (1980), *Lett. Nuovo Cimento* **28,** 476.

Ram, B. and **Halasa, R.** (1979), *Lett. Nuovo Cimento* **26,** 551.

الطيف الكتلي للهادرونات ثقيلة الكوارك

محمد رياض عرفه و عفاف باعبد الله

قسم الفيزياء ، كلية العلوم ، جامعة الملك عبد العزيز ، جدة – المملكة العربية السعودية

المستخلص. في هذا البحث قمنا بحساب كتلة طيف نظام الكواركونيوم ابسلون Υ ، و الذي تم اكتشافه في رنين تفاعلات الإلكترون و البوزترون عند طاقات معينة.

هذا النظام هو عبارة عن مستويات طاقة محصورة يتكون من الكوارك b (bottom quark) والكوارك المضاد له $\bar{b}$ (anti-bottom quark) ويطلق على هذا النظام اسم (بوتمونيم) . مستوى الطاقة الأول يعبر عن الجسيم (الميزون) Υ ، و مستويات الطاقة المثارة العليا تعبر عن الجسيمات أو الميزونات الأخرى الموجودة في هذا النظام Υ′ ، Υ″ ،....و هكذا.

يعتبر الميزون Υ جسيم ثقيل نسبيا حيث أن كتلته تساوي تقريباً ٩,٤٦ جيجا إلكترون فولت، ولذلك السبب استخدمنا طريقة WKB لإيجاد القيم الخاصة للطاقة لهذا النظام.

في هذا البحث افترضنا نموذجين مختلفين للجهد بين الكواركين في نظام ابسلون في معادلة ديراك، ثم قمنا بحساب مستويات الطاقة بواسطة كتابة برنامج بالكمبيوتر لعدة قيم مختلفة للعدد الكمي κ عند n = ٠ ، ١ ، ٢ و ٣.

أخيراً قارنا النتائج التي حصلنا عليها مع النتائج المعملية والتي أبدت توافقاً جيداً خصوصا عند العدد الكمي $\kappa = 1$.