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Bäcklund Transformations and their Prolongations

*M.F. EL-SABBAGH AND A.A. AHMAD
Mathematics Department, Faculty of Science,
King Abdulaziz University, Jeddah, Saudi Arabia;
and Mathematics Department, Faculty of Science, Minia University, Egypt

ABSTRACT. One of the main properties of equations admitting solitons, is that it can represented as the integrability conditions of Bäcklund maps. Some of these Bäcklund maps are ordinary Bäcklund maps and some are auto-Bäcklund transformations. In this paper the authors show, with examples, that prolongations of auto-Bäcklund transformations are again auto-Bäcklund transformations. Also prolongations of ordinary Bäcklund transformations yield again ordinary Bäcklund transformations with the same integrability conditions.

Introduction

Let us start with a system of differential equations Z of order $(h+1)$ with n independent variables and m dependent variables. This system will be considered as a subset of $J^{h+1}(M,N_1)$ where M is the space of independent variables, N_1 is the space of dependent variables and $j^{h+1}(M,N_1)$ denotes the jet bundle of order $(h+1)$. Choose the coordinate systems: for M, (x^a), $a=1,2,\dots,n$, and for N_1, (z^u), $u = 1,2,\dots\dots,m$.

To search for solutions of this system, in particular, when it is of some physical interest, say of soliton type, we are in need for solving what is called the Bäcklund problem for this system. This Bäcklund problem amounts to finding another space

N_2, the space of new dependent variables and a $\overset{\alpha}{C}$ – map

$$\psi : j^h(M,N_1) \times N_2 \longrightarrow j^1(M,N_2)$$

which acts trivially on M and also acts trivially on N_2, in the sense that

$$\alpha \circ pr_1 = \alpha \circ \psi \quad \text{and} \quad \beta \circ \psi = pr_2$$

where α and β denote the source and target maps and pr_i is the projection of a product on the i-th factor.

The integrability conditions of the map ψ are given in simple form by

$$\tilde{\partial}_a^{h+1} \psi_b^A - \tilde{\partial}_b^{h+1} \psi_a^A = 0 \tag{1}$$

where $\tilde{\partial}_a^{h+}$ is the hash operator on $j^h(M,N_1) \times N_2$ given by

$$\tilde{\partial}_a^{h+} = \frac{\partial}{\partial x^a} + z_a^u \frac{\partial}{\partial z^u} + z_{ab}^u \frac{\partial}{\partial z_b^u} + \qquad + \\ z^u_{aa_1, \cdots a_h} \frac{\partial}{\partial z^u_{a_1a_2 \cdots a_h}} + \psi_a^A \frac{\partial}{\partial y^A}$$

where (y^A), $(A = 1,2, \ldots, \dim N_2)$, is a coordinate presentation for the space N_2 and ψ_a^A are functions on $j^h(M,N_1) X N_2$ which define the map ψ[1-5].

If the integrability conditions of ψ, eqn. (1) as a subset of $j^{h+1}(M,N_1) X N_2$ is of the form $\tilde{Z} = ZXP$, $P \subset N_2$ then ψ is called ordinary Bäcklund map for the system Z[5].

If $\tilde{Z}$ and Z are the same, we say that ψ is an auto-Bäcklund transformation for the system Z and this is the type of main physical interest for the study of non-linear evolution equations admitting soliton solutions.

Prolongations

Since, solving the Bäcklund problem for a given system Z, is not the subject of this paper, we assume that, for the system Z, the Bäcklund problem is solved via the Bäcklund map

$$\psi : j^h(M,N_1) X N_2 \longrightarrow j^1(M,N_2) \ .$$

Now, by the first prolongation of the map ψ we mean the map ψ^1 were

$$\psi^1 : j^{h+1}(M,N_1) X N_2 \longrightarrow j^2(M,N_2)$$

which is compatible with ψ in such a way that

$$\begin{tikzcd} j^{h+1}(M,N_1) X N_2 \arrow[r, "\psi"] \arrow[d, "\pi_h^{h+1} X id_{N_2}"'] & j^2(M,N_2) \arrow[d, "\pi_1^2"] \\ j^h(M,N_1) X N_2 \arrow[r, "\psi"] & (M,N_2) \end{tikzcd}$$

commutes, where π_h^{h+1} is the projection of the $(h+1)$ – jet bundle to the h – jet bundle.

Thus this map ψ^1 is determined in an appropriate form by the functions

$\psi^A_{bc} = \tilde{\partial}^{h+1} (b\, \psi^A_c)$, ( ) denotes symmetrization between the indices b and c.

In general, the map $\psi^s : j^{h+s}(M,N_1) \times N_2 \longrightarrow j^{s+1}(M,N_2)$ which is compatible with ψ is called the *sth* prolongation of ψ and is given via the functions

$$\psi^A_{b_1 b_2 \cdots b_{s+1}} = \tilde{\partial}^{h+s} (b_1\, \tilde{\partial}^{h+s}_{b_2} \cdots \tilde{\partial}^{h+s}_{b_s})\, \psi^A_{b_1 b_2 \cdots b_{s+1}}$$

The integrability conditions of ψ^s, denoted by $\tilde{Z}^s$ are in the form of the system

$$\tilde{\partial}^{h+1}_{a_1} \cdots \tilde{\partial}^{h+s}_{a_{r-2}}\, \tilde{\partial}^{h+s} \quad (a_{r-1} \tilde{\pi}^{h+s*} \quad \psi^A_{a_r}) = 0$$

with r taking all values $2,3 \ldots\ldots, s+1$ successively. Note that $r = 2$, yields the integrability conditions for ψ itself.

Now, if there is a least integer s such that the image $\psi^s/\tilde{Z}^s$ is the system of differential equation Z' on $j^{s+1}(M,N_2)$, then the correspondence between Z and Z' is called the Bäcklund transformation determined by the Bäcklund map ψ.

If $j^o(M,N_1)$ and $j^o(M,N_2)$ are related by the identity diffeomorphism

$$(id)^o : j^o(M,N_1) \longrightarrow j^o(M,N_2)$$

and the map ψ determines a Bäcklund transformation between Z on $j^{h+1}(M,N_1)$ and Z' on $j^{h+1}(M,N_2)$ in such a way that $(id)^{h+1}(Z) = Z'$ then ψ is called a Bäcklund automorphism and the transformation is called Bäcklund self-transformation[6,7].

Now, the question is, if one starts with a Bäcklund map ψ for a system Z, which is ordinary, do its prolongations yield ordinary Bäcklund maps and if so do these prolongations have different integrability conditions than the original map ψ. Also, if one starts with a map which is auto-Bäcklund for the system Z, will its prolongations be auto-Bäcklund and what about their integrability conditions.

In fact, it is difficult to get an answer with this general framework. Also, because there is not a unified from for soliton systems so far, with which one may have a unified from of Bäcklund transformations. Therefore, we shall study some concrete examples at hand and then conclude an answer.

Some Examples

Here we give some examples of known soliton equations and calculate the prolongations of their Bäcklund transformations.

1. The sine-Gordon equation, given by $z_{12} = \sin z$, has the Bäcklund map that was originally constructed by Bäcklund himself[8,9], in the form

$$y_1 = \psi_1 = z_1 + 2a \sin \frac{1}{2}(y + z)$$

$$y_2 = \psi_2 = -z_2 - 2a^{-1} \sin \frac{1}{2}(y - z)$$

where a is a non-zero parameter.

The integrability conditions of this map give the sine-Gordon equation. Now the 1st prolongation of this map is given by:

$$y_{11} = \psi_{11} = 2[z_{11} + 2az_1 \cos \frac{1}{2}(y + z) + a^2 \sin (y + z)]$$

$$y_{12} = \psi_{12} = -2 \sin y$$

$$y_{22} = \psi_{22} = 2[-z_{22} + 2a^{-1} z_2 \cos (y + z) + a^{-2} \sin (y - z)]$$

The integrability conditions of the 1st prolonged map after some elaborate calculations give the equation $z_{12} = \sin z$. Therefore, this is an example of an auto-Bäcklund map which produces again auto-Bäcklund maps through prolongations.

2. The K. dV. equation: $z_{111} + z_2 + 12zz_1 = 0$ has the Bäcklund map discovered by Wahlquist and Estabrook,[9,10] which is given by

$$y_1 = \psi_1 = -2z - y^2$$

$$y_2 = \psi_2 = 8z^2 + 4zy^2 + 2z_{11} - 4yz_1$$

This map is an auto-Bäcklund map for the K. dV. equation. The 1st prolongation of this map is given by:

$$y_{11} = \psi_{11} = -4z_1 + 8yz + 4y^3$$

$$y_{12} = \psi_{12} = 24zz_1 + 16y^2 z_1 - 32yz^2 - 16zy^3 - 8yz_{11} - 2z_2 + 2z_{111}$$

$$y_{12} = \psi_{12} = 24zz_1 + 16y^2 z_1 - 32yz^2 - 16zy^3 - 8yz_{11} - 2z_2 + 2z_{111}$$

$$y_{22} = \psi_{22} = 32zz_2 + 8y^2 z_2 + 128yz^3 + 64z^2 y^3 - 96zz_1 y^2 + 32zyz_{11} + 4z_{112} - 8yz_{12} - 64z^2 z_1 - 16z_1 z_{11} + 32yz_1^2$$

Doing elaborate calculations, we get the integrability conditions for the prolonged map. This turns out to give again the K. dV. equation

$$z_{111} + z_2 + 12zz_1 = 0$$

As this map is known to give self-Bäcklund transformations, its prolongations will also be auto-Bäcklund maps and therefore will lead also to self-Bäcklund transformations.

3. For the equation $z_{12} = \exp z$, the Bäcklund map is given by,[5,11]

$$y_1 = \psi_1 = z_1 + \beta \exp \frac{1}{2}(y + z)$$

$$y_2 = \psi_2 = -z_2 - 2\,ß^{-1} \exp \frac{1}{2}(y - z)$$

where ß is a non-zero parameter.

The integrability condition for this map is given by $z_{12} = 0$. The *1st* prolongation is given by

$$y_{11} = \psi_{11} = z^{11} + ß\, z_1 \exp(y + z) + \frac{1}{2} ß^2 \exp(y + z)$$

$$y_{12} = \psi_{12} = \psi_{21} = \psi_{21} = -\exp y$$

$$y_{22} = \psi_{22} = -z_{22} + 2\,ß^{-1} z_2 \exp \frac{1}{2}(y - z) + 2\,ß^{-2} \exp(y - z)$$

Note that the image of ψ^1 is the zero set of $y_{12} + \exp y = 0$, which shows that this Bäcklund map ψ gives an ordinary Bäcklund transformation.

But the integrability condition of the *1st* prolonged map ψ^1 leads to the equation

$$z_{12} = 0$$

which again leads to ordinary Bäcklund transformations.

Conclusion

From the preceding examples, one can conclude that prolongations of ordinary Bäcklund transformations of differential equations give again ordinary Bäcklund transformations and also prolongations of auto-Bäcklund transformations yield again auto-Bäcklund transformations for the same system of differential eqations.

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تحـــويلات باكلونـــد وتنميـــاتها

مصطفى ف. الصبـــاغ و عادل أ. أحمـــد
قسم الرياضيات ، كلية العلوم ، جامعة الملك عبد العزيز ، جـــدة ، المملكة العربية السعودية
وقسم الرياضيات ، كلية العلوم ، جامعة المنيا - جمهورية مصر العربية

من الخواص الرئيسة للمعادلات التي لها حلول من نوع الصوليتون ، أنها يمكن كتابتها كشروط قابلية التكامل لتحويلات باكلوند العادية وبعضها يكون تحويلات باكلوند الخارجية .

وفي هذا البحث يبـيّن الباحثان مع الأمثلة أن التنمية الهندسية التفاضلية لتحويلات باكلونـد الخارجية تعطي مرة أخرى تحويلات باكلوند خارجية . كما أن التنمية الهندسية التفاضلية لتحويلات باكلوند العادية تنتج من جديد تحويلات باكلوند عادية والتي يكون لها نفس شروط قابلية التكامل .