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Modeling of a Two-Stage Merge Production with Buffer Storage

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ABSTRACT. In this article, an automated production system with two unreliable stages and a buffer is considered where the first stage has two identical machines in parallel and the second stage has a single machine. A stochastic model, which consists of a set of differential equations, is developed and solved by using a systematic matrix approach to determine line efficiency.

1. Introduction

Much interest has been shown recently to the study of the effects of machine failures and storage capacities on the efficiency of automated production systems. Many investigators have considered serial production lines with finite capacity buffers between stages and attempted to obtain exact solutions. Exact solutions have been obtained for two and three-stage lines with unreliable machines. Buzacott[1-3], Elsayed and Turley[4], Gershwin and Berman[5], Okamura and Yamashina[6], Savsar and Biles[7], Sheskin[8], Soyster and Toff[9], Wijngaard[10], and Malathronas *et al.*[11] have considered two-stage serial production lines with intermediate buffers and obtained analytical solutions under different assumptions. Gershwin and Schick[12] obtained exact solutions for three-stage lines. They also presented a comprehensive literature review on the subject. Ignall and Silver[13] obtained approximate solutions for a two stage line with N machines in each stage. Mitra[14] developed analytical models for two-stage fluid flow lines. Several other related models are given in the references[15-20].

All of the above mentioned models, with the exception of Ignall and Silver[13] and Mitra[14], assume a single machine in each stage with a finite buffer between stages. This paper presents an analytical solution for a production system in which the output of two parallel machines flows into a third machine through a finite capacity buffer. Figure 1 illustrates the production system considered here. It is not an assembly system, but a serial production line with merge configuration. Since many real life production systems are of this nature, the model presented here may find useful applications. In particular, the model is applicable to computer controlled manufacturing systems with automated handling equipment, robots, and buffers. The decision of how to allocate the buffer to improve the production rate is of practical importance. The need for buffer storage for the design of computer systems is also essential; for example, a buffer storage is required for: (i) The storage of information prior to output and followig input from auxiliary storage or input/output devices, (ii) The storage of data and messages communicated between active tasks, (iii) The storage of messages arriving from or being transmitted to remote terminals such as in a time-sharing system[4].

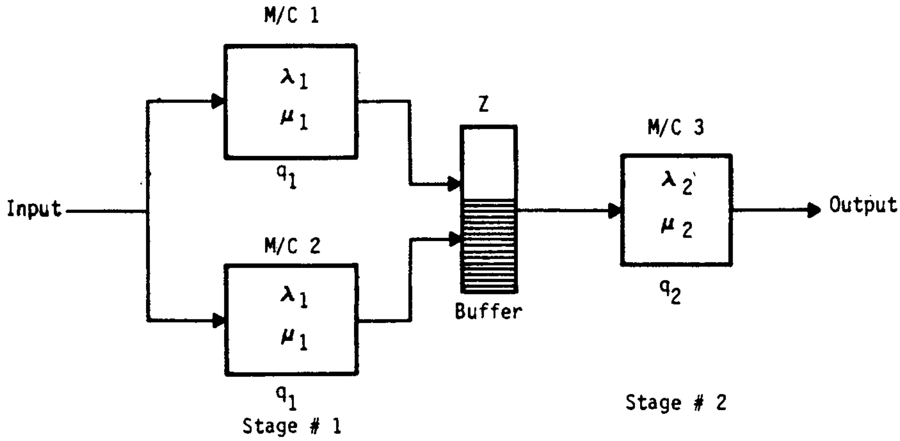

FIG. 1. A two-stage production line with merge configuration.

2. System Operation and Assumptions

The production system under consideration is assumed to have the following operation characteristics.

1. Machines $M1$ and $M2$, in the first stage, are identical and have the same failure, repair, and production rates denoted as λ_1, μ_1 and q_1 respectively.
2. Machine $M3$, in the second stage, has a failure rate of λ_2, repair rate of μ_2, and a production rate of q_2 which is assumed as $q_2 = 2q_1$.
3. The buffer has a finite capacity for z units.
4. There is one repairman for each stage.
5. Machine failures and repairs are random and are exponentially distributed.

Mean values or rates for these random variables can be determined on the basis of statistical investigation. In almost all of the previous studies given in the references, exponential distribution has been used to model equipment failures and repairs in production lines. This is mainly due to the fact that this distribution can actually model most of the real life situations and is easy to work with.

6. Failure rate of machine $M3$ reduces to $\lambda_2' = \lambda_2/2$ when its operation rate is reduced to q_1 from $2q_1$, *i.e.*, failures are operation dependent.

7. The system operates until a machine fails. If $M1$ (or $M2$) fails, the operation continues until one of the following events occurs: i) The buffer level is reduced to zero by machine $M3$; ii) Machine $M3$ fails; or iii) machine $M2$ (or $M1$) fails. If the buffer level reduces to zero before the repair of failed machine is completed, the second stage ($M3$) slows down and operates at rate q_1 instead of its normal rate q_2. If both machines, $M1$ and $M2$ fail, the second stage continues operation until the buffer is empty, at which time, $M3$ is forced down due to unavailability of incoming parts. The failure of $M3$ will force $M1$ and/or $M2$ down, *i.e.*, blocked, when the buffer reaches its maximum level z. In any case, a forced down machine will not fail.

The following notations are used to describe the state of production system with above characteristics and operation policy.

- $S_{ijx}(t, x)$ State of the system at time t with buffer size x, $t > 0$, $0 \leq x \leq z$, and i, j machines operating at the first and second stage respectively ($i = 0, 1, 2$ and $j = 0, 1$).
- x Buffer size, $0 \leq x \leq z$.
- z Maximum buffer capacity.
- $f_{ijx}(t, x)$ Probability distribution function of state $S_{ijx}(t, x)$ with $i = 0, 1, 2; j = 0, 1$; $0 < x < z$, (x variable).
- $P_{ijx}(t)$ Probability of system state S_{ijx} with x constant at either $x = 0$, or $x = z$; $i = 0, 1, 2; j = 0, 1$.

For those states in which buffer level varies, *i.e.*, $0 < x < z$, the system changes its state with respect to buffer level x as well as the time t. There are six such states, namely; $S_{00x}(t, x)$, $S_{01x}(t, x)$, $S_{11x}(t, x)$, $S_{21x}(t, x)$, $S_{20x}(t, x)$, $S_{10x}(t, x)$. For those states in which buffer is either empty or full, the system changes its state with respect to time t only. There are also six such states and they are $S_{010}(t, 0)$, $S_{110}(t, 0)$, $S_{210}(t, 0)$, $S_{10z}(t, z)$, $S_{20z}(t, z)$, $S_{21z}(t, z)$. Note that the probability of two machines failing at the same time, while the buffer is full or empty, is assumed to be zero since such an event has an infinitely small probability.

The marginal probabilities for the first six states are given by

$$P_{ijx}(t) = {}_0\!\int^z f_{ijx}(t, x)dx \qquad \begin{array}{l} i = 0, 1, 2 \\ j = 0, 1 \\ 0 < x < z \end{array}$$

3. The Stochatic Model

Using the notation above, the functioning of the production system under consideration can be described by means of the transition flow diagram given in Fig. 2, where the dashed lines indicate boundry state transitions and dark solid lines indicate the transitions that cause boundry conditions.

For notational convenience, let us use f_{ijx} to denote $f_{ijx}(t, x)$. From the transition flow diagram, given in Fig. 2, it is possible to obtain a set of differential equations that govern the production system described above. There are twelve state equations, (1) and (12), corresponding to the twelve system states, as given below.

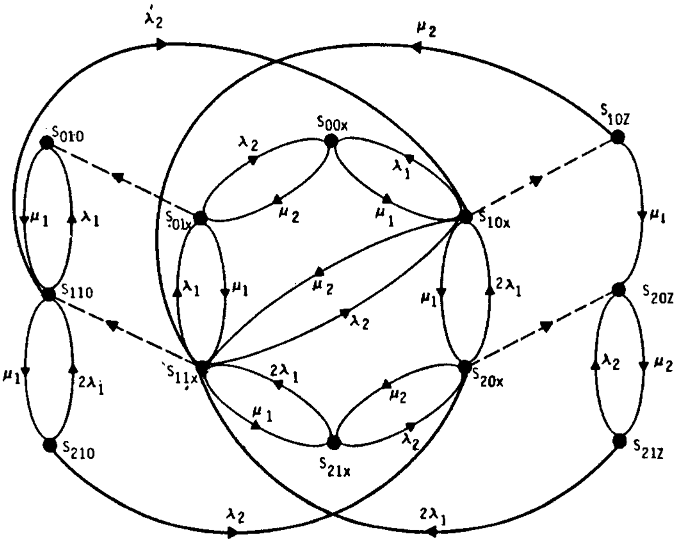

FIG. 2. Probability transition flow diagram.

$$\frac{\partial f_{00x}}{\partial t} = -(\mu_1 + \mu_2) f_{00x} + \lambda_2 f_{01x} + \lambda_1 f_{10x} \tag{1}$$

$$\frac{\partial f_{21x}}{\partial t} = -(2\lambda_1 + \lambda_2) f_{21x} + \mu_1 f_{11x} + \mu_2 f_{20x} \tag{2}$$

$$\frac{\partial f_{01x}}{\partial t} - q_2 \frac{\partial f_{01x}}{\partial x} = \mu_2 f_{00x} - (\mu_1 + \lambda_2) f_{01x} + \lambda_1 f_{11x} \tag{3}$$

$$\frac{\partial f_{11x}}{\partial t} - q_1 \frac{\partial f_{11x}}{\partial x} = 2\lambda_1 f_{21x} + \mu_1 f_{01x} - (\lambda_1 + \lambda_2 + \mu_1) f_{11x} + \mu_2 f_{10x} \tag{4}$$

$$\frac{\partial f_{10x}}{\partial t} + q_1 \frac{\partial f_{10x}}{\partial x} = \mu_1 f_{00x} + \lambda_2 f_{11x} - (\mu_1 + \mu_2 + \lambda_1) f_{10x} + 2\lambda_1 f_{20x} \tag{5}$$

$$\frac{\partial f_{20x}}{\partial t} + 2q_1 \frac{\partial f_{20x}}{\partial x} = \lambda_2 f_{21x} + \mu_1 f_{10x} - (\mu_2 + 2\lambda_1) f_{20x} \tag{6}$$

$$\frac{\partial p_{010}}{\partial t} = - \mu_1 p_{010} + \lambda_1 p_{110} + q_2 f_{01x}(0) \tag{7}$$

$$\frac{\partial p_{110}}{\partial t} = \mu_1 p_{010} - (\lambda_1 + \mu_1 + \lambda_2') p_{110} + 2\lambda_1 p_{210} + q_1 f_{11x}(0) \tag{8}$$

$$\frac{\partial p_{10z}}{\partial t} = - (\mu_1 + \mu_2) p_{10z} + q_1 f_{10x}(z) \tag{9}$$

$$\frac{\partial p_{20z}}{\partial t} = \mu_1 p_{10z} - \mu_2 p_{20z} + \lambda_2 p_{21z} + 2q_1 f_{20x}(z) \tag{10}$$

$$\frac{\partial p_{210}}{\partial t} = \mu_1 p_{110} - (2\lambda_1 + \lambda_2) p_{210} \tag{11}$$

$$\frac{\partial p_{21z}}{\partial t} = \mu_2 p_{20z} - (2\lambda_1 + \lambda_2) p_{21z} \tag{12}$$

The boundry conditions are caused by the flows from states S_{110}, S_{210}, S_{10z}, and S_{21z} to the states S_{10x}, S_{20x}, S_{11x} and S_{11x} respectively. These conditions are stated as follows

$$\lambda_2' p_{110} = q_1 f_{10x}(0) \tag{13}$$

$$\lambda_2 p_{210} = 2q_1 f_{20x}(0) \tag{14}$$

$$\mu_2 p_{10z} = q_1 f_{11x}(z) \tag{15}$$

$$2\lambda_1 p_{21z} = q_1 f_{11x}(z) \tag{16}$$

Equations (1) to (6), which are decoupled from equations (7) to (12), can be represented in matrix notations as follows

$$[\dot{F}]_t + [q_l]\,[\dot{F}]_x = [A]\,[F] \tag{17}$$

where, $[\dot{F}]_t = \frac{\partial}{\partial t} \begin{vmatrix} F_1 \\ \cdots \\ F_2 \end{vmatrix}$, $[\dot{F}]_x = \frac{\partial}{\partial x} \begin{vmatrix} F_1 \\ \cdots \\ F_2 \end{vmatrix}$

$$[F] = \begin{vmatrix} F_1 \\ \cdots \\ F_2 \end{vmatrix}, \quad [F_1] = \begin{vmatrix} f_{00x}(t, x) \\ f_{21x}(t, x) \end{vmatrix}, \quad [F_2] = \begin{vmatrix} f_{01x}(t, x) \\ f_{11x}(t, x) \\ f_{10x}(t, x) \\ f_{20x}(t, x) \end{vmatrix}$$

$$[q_l] = \left|\begin{array}{cc:cccc} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & -q_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & -q_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & q_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2q_1 \end{array}\right| = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & q_{01} \end{array}\right]$$

$$[A] = \left|\begin{array}{cc:cccc} -(\mu_1+\mu_2) & 0 & \lambda_2 & 0 & \lambda_1 & 0 \\ 0 & -(2\lambda_1+\lambda_2) & 0 & \mu_1 & 0 & \mu_2 \\ \hdashline \mu_2 & 0 & -(\mu_1+\lambda_2) & \lambda_1 & 0 & 0 \\ 0 & 2\lambda_1 & \mu_1 & -(\mu_1+\lambda_1+\lambda_2) & \mu_2 & 0 \\ \mu_1 & 0 & 0 & \lambda_2 & -(\mu_1+\mu_2+\lambda_1) & 2\lambda_1 \\ 0 & \lambda_2 & 0 & 0 & \mu_1 & -(\mu_2+2\lambda_1) \end{array}\right| = \left|\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right|$$

At steady state, the partial derivatives with respect to time t approach zero, *i.e.*, $[\dot{F}]_t = 0$, and $[F_1]$ & $[F_2]$ are now functions of x only. The new system of differential equations is written in matrix notations as follows

$$[A_1]\,[F_1] + [A_2]\,[F_2] = 0 \tag{18}$$

$$[A_3]\,[F_1] + [A_4]\,[F_2] = [q_{01}]\,[\dot{F}_2]_x \tag{19}$$

Substituting $[F_1]$ from (18) into (19), the following equation set is obtained :

$$[\dot{F}_2] = [\Omega]\,[F_2] \tag{20}$$

where, $[\Omega] = [q_{01}]^{-1}\ \{[A_4] - [A_3]\,[A_1]^{-1}\,[A_2]\}$

Equation set (20), which constitutes a system of homogeneous differential equations, has the following general solution[21].

$$[F_2] = [S]\,[e^{kx}]\,[C] = [\psi_2]\,[C] \tag{21}$$

$[S]$ is a 4×4 matrix containing the eigenvectors of matrix $[\Omega]$, and $[e^{kx}]$ is a 4×4 diagonal matrix with $e^{k_i x}$ in the i *th* diagonal; where k_i is the i *th* eigenvalue of $[\Omega]$.

$[C] = (c_1, c_2, c_3, c_4)^T$, constant coefficients to be determined by the initial conditions.

By substituting $[F_2]$ from equation set (21) into equation set (18), $[F_1]$ is determined as follows

$$[F_1] = -[A_1]^{-1}\ [A_2]\,[S]\,[e^{kx}]\,[C] = [\Psi_1]\,[C] \tag{22}$$

Now, equations (7) to (12), which constitute a linear system, can be represented as follows

$$[\dot{P}]_t = [B]\,[P]_t + [q_{ll}]\,[F_0] \tag{23}$$

where,

$$[\dot{P}]_t = \frac{\partial}{\partial t}[P]\ ,\ [P] = \left|\begin{array}{c} P_3 \\ \cdots \\ P_4 \end{array}\right|\ ,\ [P_3] = \left|\begin{array}{c} p_{010}(t) \\ p_{110}(t) \\ p_{10z}(t) \\ p_{20z}(t) \end{array}\right|\ ,\ [P_4] = \left|\begin{array}{c} p_{210}(t) \\ p_{21z}(t) \end{array}\right|$$

$$[B] = \left|\begin{array}{cccc:cc} -\mu_1 & \lambda_1 & 0 & 0 & 0 & 0 \\ \mu_1 & -(\lambda_1+\mu_1+\lambda_2') & 0 & 0 & 2\lambda_1 & 0 \\ 0 & 0 & -(\mu_1+\mu_2) & 0 & 0 & 0 \\ 0 & 0 & \mu_1 & -\mu_2 & 0 & \lambda_2 \\ \hdashline 0 & \mu_1 & 0 & 0 & -(2\lambda_1+\lambda_2) & 0 \\ 0 & 0 & 0 & \mu_2 & 0 & -(2\lambda_1+\lambda_2) \end{array}\right| = \left|\begin{array}{c:c} B_1 & B_2 \\ \hdashline B_3 & B_4 \end{array}\right|$$

$$[q_{ll}] = \left|\begin{array}{cccc:cc} q_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & q_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & q_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2q_1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right| = \left|\begin{array}{c:c} q_{02} & 0 \\ \hdashline 0 & 0 \end{array}\right|,\ [F_0] = \left|\begin{array}{c} f_{01z}(0) \\ f_{11z}(0) \\ f_{10z}(z) \\ f_{20z}(z) \\ \hdashline 0 \\ 0 \end{array}\right| = \left|\begin{array}{c} F_{01} \\ \hdashline 0 \end{array}\right|$$

For the steady state solution, $[\dot{P}]_t = 0$, and therefore

$[B_1]\,[P_3] + [B_2]\,[P_4] + [q_{02}]\,[F_{01}] = 0$ and $[B_3]\,[P_3] + [B_4]\,[P_4] = 0$ which are solved to obtained $[P_3]$ and $[P_4]$ as

$$[P_3] = [D]\,[C] \tag{24}$$

$$[P_4] = [H]\,[C] \tag{25}$$

where, $[D] = -\{[B_1]-[B_2]\,[B_4]^{-1}\,[B_3]\}^{-1}\,[q_{02}]\,[R]$

and $[H] = -\,[B_4]^{-1}\,[B_3]\,[D]$, where

$$[R] = \left|\begin{array}{cccc} s_{11} & s_{12} & s_{13} & s_{14} \\ s_{21} & s_{22} & s_{23} & s_{24} \\ s_{31}e^{k_1 z} & s_{32}e^{k_2 z} & s_{33}e^{k_3 z} & s_{34}e^{k_4 z} \\ s_{41}e^{k_1 z} & s_{42}e^{k_2 z} & s_{43}e^{k_3 z} & s_{44}e^{k_4 z} \end{array}\right|$$

s_{ij} are the elements of matrix $[S]$.

Equations (21), (22), (24) and (25) give the complete solution for the state probabilities if the constant coefficients, $[C]$, are determined. In order to determine $[C]$, boundary conditions, the normalizing condition, and some matrix manipulations are needed. The procedure for obtaining C is presented in the appendix.

The constant coefficients, $[C]$ are substituted into equation sets (21), (22), (24), and (25) to obtain $[F_1]$, $[F_2]$ $[P_3]$ and $[P_4]$. $[P_2]$ and $[P_1]$ are obtained by integrating $[F_2]$ and $[F_1]$ respectively. The results are given in the appendix by equation sets

(A5) and (A6). $[P_1]$, $[P_2]$, $[P_3]$ and $[P_4]$ are the steady state probabilities for twelve states of the system as described before. Combining all these into a single vector P_s, the steady state probability vector can be written as follows

$$[P_s] = [p_{00x}, p_{21x}, p_{01x}, p_{11x}, p_{10x}, p_{20x}, p_{010}, p_{110}, p_{10z}, p_{20z}, p_{210}, p_{21z}]$$

These probabilities can be used to study the steady state behaviour of the line.

In the model presented here, it is assumed that $2q_1 = q_2$, *i.e.*, the system is balanced. For $2q_1 \neq q_2$, a state transition would occur from S_{21x} to S_{21z} if $2q_1 > q_2$, and a transition would occur from S_{21x} to S_{210} if $2q_1 < q_2$. Each case must be treated separately by changing the differential equation given by (2) and adding an appropriate boundary condition as follows

$$\frac{\partial f_{21x}}{\partial t} + \Delta q \frac{\partial q f_{21x}}{\partial x} = -(2\lambda_1 + \lambda_2) f_{21x} + \mu_1 f_{11x} + \mu_2 f_{20x}$$

$$2\lambda_1 P_{21z} = \Delta q f_{21z} \quad \text{for } 2q_1 > q_2$$

$$\lambda_2 P_{210} = -\Delta q f_{21z} \quad \text{for } 2q_1 < q_2$$

where $\Delta q = 2q_1 - q_2$

4. Production System Efficiency

One can view the system efficiency, ε, as the proportion of time that the last stage is in operation. In this model however, there are two possible production rates for the last stage, namely normal rate q_2 and reduced rate q_1.

Proportion of time that the last stage is producing parts at its full rate q_2 is given by

$$\varepsilon_1 = \sum_{i=0}^{2} p_{i1x} + p_{210} + p_{21z}$$

and the proportion of time that the last stage is producing parts at the reduced rate, q_1, is given by $\varepsilon_2 = p_{110}$

where p_{ijx} are the state probabilities as defined before.

Line efficiency is not merely the sum of ε_1 and ε_2. The actual production rate, β_ℓ must be considered in calculating line efficiency.

$$\beta_\ell = q_2\varepsilon_1 + q_1\varepsilon_2$$

Thus the actual line efficiency $\varepsilon = \beta_\ell / q_2$.

Efficiency of the first stage in the line is calculated as follows.

Let $\alpha_1 = p_{21x} + p_{210} + p_{20x}$, proportion of time the first stage is producing at rate $2q_1$. $\alpha_2 = p_{110} + p_{11x} = p_{10x}$, proportion of time the first stage is producing at rate q_1. The efficiency of the first stage is thus $\alpha_1 + \alpha_2/2$. Efficiency of the second stage, which produces the final product, is considered as the system efficiency as calculated above.

5. Computational Results

The procedure for computing the steady state probabilities presented above was programmed on a VAX Mainframe Computer with the IMSL routines used for matrix operations. Several runs were made to determine the production line efficiency under different cases. The computation time was very small and did not require a particular attention.

The results of several runs made for four different cases are presented in Fig. 3-6. In case 1, the failure rates of machines, λ_1 and λ_2, were equal at $\lambda_1 = \lambda_2 = 0.4$ failures/unit time. In case 2, $\lambda_1 > \lambda_2$ at $\lambda_1 = 0.4$ and $\lambda_2 = 0.2$; in case 3, $\lambda_1 < \lambda_2$ at $\lambda_1 = 0.2$ and $\lambda_2 = 0.4$; and in case 4, $\lambda_1 = \lambda_2 = 0.2$ failures/time unit. In all four cases, the repair rates μ_1 and μ_2 were changed from 2 to 4 as shown in the figures. The production rates of the machine in the first stage were $q_1 = 5$ units/hour while the production rate in the second stage was $q_2 = 10$ units/hour. Buffer capacity was changed from 0 to 30 units. In each case, the line efficiency was calculated using the equation given for ε above.

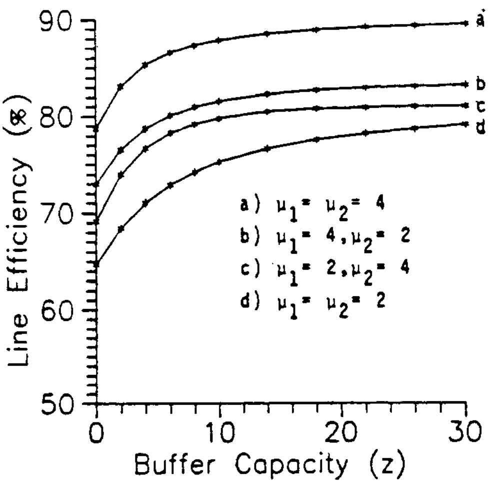

FIG. 3. Line efficiency for case 1 ($\lambda_1 = 0.4$, $\lambda_2 = 0.4$).

The results indicated that the buffer capacity had significant effect on increasing the line efficiency for up to approximately $z = 10$ units in almost all cases. Thereafter, the effect was not as high. For fixed failure rates, as the repair rate was increased, the line efficiency also increased significantly for all buffer capacities. The main reason for this is obviously due to the fact that as the repair rate is increased, the equipment down time is reduced and thus the line efficiency is increased. However, because of unavoidable random failures, which depend on machine failure rates, the line efficiency could only be increased up to a certain level with additional buffer capacity. Thereafter, increasing buffer capacity would not increase the line efficiency significantly.

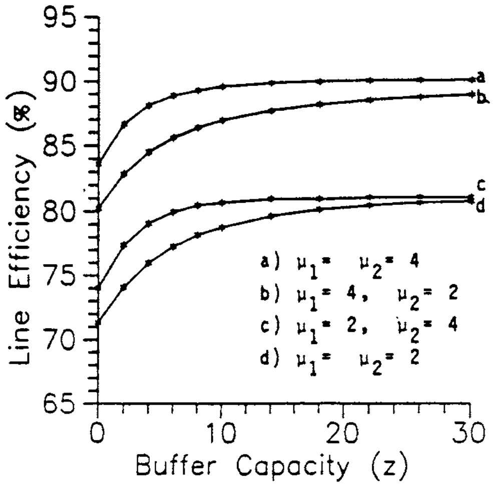

FIG. 4. Line efficiency for case 2 ($\lambda_1 = 0.4$, $\lambda_2 = 0.2$).

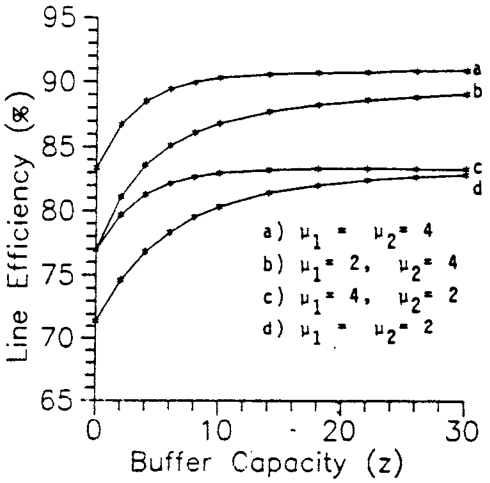

FIG. 5. Line efficiency for case 3 ($\lambda_1 = 0.2$, $\lambda_2 = 0.4$).

6. Conclusion

In this paper, methodology has been developed to obtain the steady state line efficiency for a production line consisting of two stages with a merge configuration and a finite intermediate buffer.

The results and the model presented here signify that a line should be designed after some investigation of the effects of buffer capacity, along with failure/repair profiles, on the line efficiency. Such analysis can help the design engineers and operation managers to assess and to improve the efficiency of a production line under consideration.

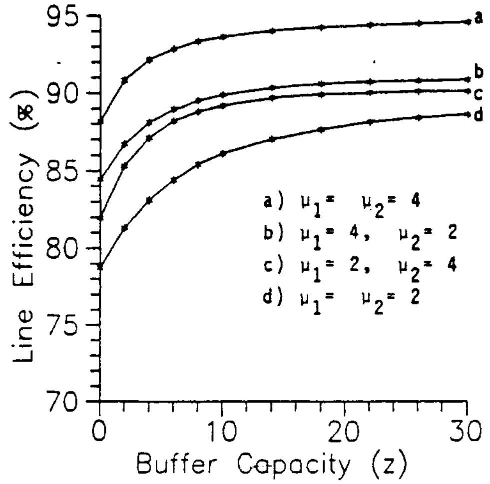

FIG. 6. Line efficiency for case 4 ($\lambda_1 = 0.2$, $\lambda_2 = 0.2$).

The model presented here gives exact solutions to the extended buffer storage problem considered in the literature. One of the main features of this model is that the exact solution is not restricted by the buffer capacity, *z*. In most of the previous models however, the state space depends on the buffer size and thus the problems could only be solved for up to a certain buffer size, *z*. Furthermore, the previous models give only approximate solutions, particularly for the production lines with merge configuration. Here, an exact solution is obtained based on a systematic matrix approach that lends itself to easy computation. Therefore, it is possible to extend the model presented here further to obtain exact solutions for a general two-stage line with *N* machines in each stage and for the same line configuration with a single repair crew for the whole line. Work is currently in progress for these extensions.

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Appendix

Determination of Constant Coefficients, [*C*]

The constant coefficients, which are needed to solve the differential equations given by equation sets (21), (22), (24) and (25), are obtained in the following steps.

Step 1

From the boundary conditions given by equations (13) - (15), the following system of equations is formed. The last boundary condition, equation (16), is not needed since it is redundant.

$$(p_{110}, p_{210}, p_{10z})^T = [\Phi][C] \tag{A1}$$

where

$$[\Phi] = q_1 \cdot \begin{vmatrix} \dfrac{s_{31}}{\lambda_2'} & \dfrac{s_{32}}{\lambda_2'} & \dfrac{s_{33}}{\lambda_2'} & \dfrac{s_{34}}{\lambda_2'} \\ \dfrac{2s_{41}}{\lambda_2} & \dfrac{2s_{42}}{\lambda_2} & \dfrac{2s_{43}}{\lambda_2} & \dfrac{2s_{44}}{\lambda_2} \\ \dfrac{s_{21}e^{k_1 z}}{\mu_2} & \dfrac{s_{22}e^{k_2 z}}{\mu_2} & \dfrac{s_{23}e^{k_3 z}}{\mu_2} & \dfrac{s_{24}e^{k_4 z}}{\mu_2} \end{vmatrix}$$

and $[C]$ is the vector of constant coefficients as defined before.

Step 2

From equations (24) and (25), p_{110}, p_{210}, and p_{10z} are extracted and a new system of equations is constructed as follows

$$\begin{vmatrix} p_{110} \\ p_{210} \\ p_{10z} \end{vmatrix} = \begin{vmatrix} d_{21} & d_{22} & d_{23} & d_{24} \\ h_{11} & h_{12} & h_{13} & h_{14} \\ d_{31} & d_{32} & d_{33} & d_{34} \end{vmatrix} \begin{vmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{vmatrix} = [V]\,[C] \tag{A2}$$

where d_{ij} and h_{ij} are the elements of matrices D and H respectively.

Step 3

After equating the equations (A1) and (A2), the following is obtained.

$$[Y]\,[C] = [0] \tag{A3}$$

where $[Y] = [\Phi] - [V]$ and $[0]$ is a 3×1 zero vector.

Equation set (A3) gives the first three equations to determine the four constant coefficients, $[C]$. The fourth equation is obtained from the normalizing condition as given in the following step.

Step 4

The normalizing condition, which states that the sum of state probabilities must equal one, is used as the fourth equation in the system given by equation set (A3). This condition can be written and simplied as follows

$$\sum_{i=0}^{2} \sum_{j=0}^{1} p_{ijx} + \sum_{i=0}^{2} p_{i10} + p_{10z} + p_{20z} + p_{21z} = 1$$

where $p_{ijx} = {}_0\!\int^z f_{ijx}(x)\,dx$

thus,

$$\sum_{i=0}^{2} \sum_{j=0}^{1} \int_0^z f_{ijx}(x)\,dx + \sum_{i=0}^{2} p_{i10} + p_{10z} + p_{20z} + p_{21z} = 1 \tag{A4}$$

In order to obtain the above equation, it is necessary to integrate equation sets (21) and (22).

Integration of equation set (22) gives the following.

$$[P_2] = [\psi_2']\,[C] \tag{A5}$$

where, $[\psi_2'] = [S]\,[\Lambda]$

where $[\Lambda]$ is a 4×4 diagonal matrix with $(e^{k_i z} - 1)/k_i$ in the *i th* diagonal, and integration of equation set (22) results in the following.

$$[P_1] = [\psi_1']\,[C] \tag{A6}$$

where, $[\psi_1'] = -[A_1]^{-1}\,[A_2]\,[S]\,[\Lambda]$

Finally, sum of the rows of matrices, $[D]$, $[H]$, $[\psi_1']$ and $[\psi'_2]$ are equated to 1 to obtain equation (A4).

Step 5

Equation (A4) is appended to the set (A3) as the fourth equation and a new system of four equations in four unknowns, $[C]$, is obtained.

$$[Y'][C] = [G] \tag{A7}$$

where, $[G] = (0\ 0\ 0\ 1)^T$ and $[Y']$ is $[Y]$ with equation (A4) appended.

Step 6

The set of equations given in (A7) is solved to obtained the unknown constant coefficients $[C]$.

$$[C] = [Y']^{-1}[G]$$

النمـذجـة لعمليـة إنتاجيـة مكونـة من مرحلتـين تنـدمجان مع وجـود مخـزون

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المستخلص . في هذا المقال ، دراسة لمرحلتين غير موثوق بهما يكوّنان خط إنتاج أُتوماتي مع اعتبـار أن هنـاك مخزون . المرحلة الأولى مكـونـة من اثنين من المـاكينات المتماثلة على التوازي ، والمرحلة الثانية مكونة من ماكينة واحدة . وقد تم تطوير نموذج عشوائي مكون من مجموعة من المعادلات التفاضلية ، وقد تم حله باستخدام المحددات لتحديد كفاءة الخط .